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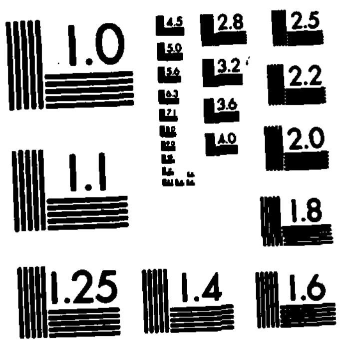

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APPLICATION OF THE NEW E-FIELD SOLUTION TO A SURFACE OF REVOLUTION

by

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A computer program is given to implement the new E-field solution for the surface density of electric current and electric charge induced on a perfectly conducting surface of revolution immersed in the electromagnetic field of a plane wave. Unlike other E-field solutions, the new E-field solution works well for a surface whose maximum dimension is less than a hundredth of a wavelength. The computer program was written for axial incidence but can easily be modified to handle oblique incidence. Results obtained by using this com-

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20. ABSTRACT (continued)

Cont

puter program agreed with results obtained by using two other computer programs given here, one for Bouwkamp's power series solution for a conducting circular disk and one for the Mie series solution for a conducting sphere.

CONTENTS

I. INTRODUCTION

Computer programs for the new E-field solution for a conducting surface of revolution are described and listed in this report. The new E-field solution was introduced in [1]. It is assumed that the reader is familiar with [1]. The conducting surface of revolution is called S. The surface S may be either open or closed. Neither fins nor wires are attached to S. If S is open, then S is assumed to be bounded by at most two contours, one at the beginning of the generating curve of S, and one at the end of the generating curve of S.

[1, Eq. (86)] can be rewritten as

$$Z_n \vec{I}_n = \vec{V}_n , \quad n = 0, \pm 1, \pm 2, \ldots \tag{1}$$

where

$$Z_n = \begin{bmatrix} Z_n^{mm} & Z_n^{me} \\ Z_n^{em} & Z_n^{ee} \end{bmatrix} \tag{2}$$

$$\vec{I}_n = \begin{bmatrix} \vec{I}_n^m \\ \vec{I}_n^e \end{bmatrix} \tag{3}$$

$$\vec{V}_n = \begin{bmatrix} \vec{V}_n^m \\ \vec{V}_n^e \end{bmatrix} \tag{4}$$

Equation (1) is called the moment equation, Z_n is called the moment matrix, and $\vec{V}_n$ is called the excitation vector. The subroutine ZMAT of Section II

calculates the elements of Z_n for n = M1, M1+1, M1+2,...M2 where M1 and M2 are non-negative integers and M2 $\geq$ M1. The subroutine ZMAT calls the function BLOG of Section III. The subroutine PLANE of Section IV calculates the elements of $\vec{V}_n$ for n = M1, M1+1, M1+2,...M2. The subroutines DECOMP and SOLVE of Section V solve (1) for $\vec{I}_n$. The subroutines ZMAT and PLANE are similar to the subroutines listed on pages 51-55 and 61-62 of [2]. The subprograms BLOG, DECOMP, and SOLVE are exactly the same as in [2].

The main program of Section VI calls the subroutines ZMAT, PLANE, DECOMP, and SOLVE in order to calculate the surface density of electric current and electric charge induced on S by a plane wave that propagates along the z axis. The z axis is the axis about which the generating curve of S is rotated. Such a plane wave is called an axially incident plane wave. For an axially incident plane wave, it suffices to solve (1) only for n=1. Because the subroutines ZMAT and PLANE are designed for n = M1, M1+1, M1+2,...M2, these subroutines are more general than the main program of Section VI.

In [1, Figs. 1,2, and 3], the new E-field solution for the current and charge on a small circular disk is compared with Bouwkamp's power series solution [3]. In Section VII, Bouwkamp's power series solution for the surface density of electric current on a conducting circular disk of unit radius excited by an axially incident plane wave is converted to mksc units [4, p. 1] for a disk of arbitrary radius a. The electric charge on the disk is extracted from this electric current. A computer program which calculates this current and charge is described and listed in Section VII.

In [1, Figs. 4,5, and 6], the new E-field solution for the current and charge on a small sphere is compared with the Mie series solution

[4, Eq. (6-103)], [5] for the surface density of electric current and electric charge on a conducting sphere excited by a plane wave. A computer program which calculates the Mie series solution for the current and charge is described and listed in Section VIII.

II. THE SUBROUTINE ZMAT

The subroutine ZMAT calculates the elements of Z_n of (2).

For n = 0, the expansion functions are given by [1, Eqs. (91)-(92)] and the testing functions are given by [1, Eqs. (93)-(94)] so that

$$\left.\begin{aligned} Z_0^{mm} &= Z_0^{\phi\phi} \\ Z_0^{em} &= Z_0^{t\phi} \\ Z_0^{me} &= Z_0^{\phi t} \\ Z_0^{ee} &= Z_0^{tt} \end{aligned}\right\} \tag{5}$$

where the elements of the matrices on the right-hand sides of (5) are given by [2, Eqs. (9)-(12)]. It is evident from [2, Eqs. (10)-(11)] that all the elements of $Z_0^{t\phi}$ and $Z_0^{\phi t}$ are zero. Therefore, for n = 0, (2) reduces to

$$Z_0 = \begin{bmatrix} Z_0^{mm} & 0 \\ 0 & Z_0^{ee} \end{bmatrix} \tag{6}$$

For n ≠ 0, the expansion functions are given by [1, Eqs. (100)-(101)] and the testing functions are given by [1, Eqs. (102)-(103)] so that

$$(Z_n^{mm})_{ij} = (Z_n^{tt})_{ij} + \alpha_{nj}(Z_n^{t\phi})_{ij} - \alpha_{n,j+1}(Z_n^{t\phi})_{i,j+1} - \alpha_{ni}(Z_n^{\phi t})_{ij} - \alpha_{ni}\alpha_{nj}(Z_n^{\phi\phi})_{ij}$$

$$+ \alpha_{ni}\alpha_{n,j+1}(Z_n^{\phi\phi})_{i,j+1} + \alpha_{n,i+1}(Z_n^{\phi t})_{i+1,j} + \alpha_{n,i+1}\alpha_{nj}(Z_n^{\phi\phi})_{i+1,j}$$

$$- \alpha_{n,i+1}\alpha_{n,j+1}(Z_n^{\phi\phi})_{i+1,j+1} \tag{7}$$

$$(Z_n^{em})_{ij} = k\rho_i((Z_n^{\phi t})_{ij} + \alpha_{nj}(Z_n^{\phi\phi})_{ij} - \alpha_{n,j+1}(Z_n^{\phi\phi})_{i,j+1}) \tag{8}$$

$$(Z_n^{me})_{ij} = k\rho_j((Z_n^{t\phi})_{ij} - \alpha_{ni}(Z_n^{\phi\phi})_{ij} + \alpha_{n,i+1}(Z_n^{\phi\phi})_{i+1,j}) \tag{9}$$

$$(Z_n^{ee})_{ij} = k^2\rho_i\rho_j(Z_n^{\phi\phi})_{ij} \tag{10}$$

where the Z_n's on the right-hand sides of (7)-(10) are given by [2, Eqs. (9)-(12)] and α_{nj} is given by [1, Eq. (99)].

$$\alpha_{nj} = \frac{j\rho_j}{n\Delta_j} \tag{11}$$

All the subscripts j in (11) coincide with each other, but the j which multiplies ρ_j on the right-hand side of (11) is $\sqrt{-1}$. It is evident from [2, Eqs. (9)-(12)] that

$$\begin{bmatrix} Z_{-n}^{tt} & Z_{-n}^{t\phi} \\ Z_{-n}^{\phi t} & Z_{-n}^{\phi\phi} \end{bmatrix} = \begin{bmatrix} Z_n^{tt} & -Z_n^{t\phi} \\ -Z_n^{\phi t} & Z_n^{\phi\phi} \end{bmatrix} \tag{12}$$

Equations (7)-(12) imply that

$$\begin{bmatrix} Z_{-n}^{mm} & Z_{-n}^{me} \\ Z_{-n}^{em} & Z_{-n}^{ee} \end{bmatrix} = \begin{bmatrix} Z_n^{mm} & -Z_n^{me} \\ -Z_n^{em} & Z_n^{ee} \end{bmatrix} \tag{13}$$

Therefore, it suffices to calculate the elements of Z_n of (2) only for non-negative values of n.

New matrices $\{\hat{Z}_n, n=0, \pm 1, \pm 2, \ldots\}$ are defined by

$$\hat{Z}_0 = \begin{bmatrix} Z_0^{ee} & 0 \\ 0 & \beta Z_0^{mm} \end{bmatrix} \tag{14}$$

$$\hat{Z}_n = \beta Z_n \ , \ n = \pm 1, \pm 2, \ldots \tag{15}$$

where

$$\beta = (\tfrac{1}{2} k\Delta_1)^{-2} \tag{16}$$

Later, it will become evident that the scale factor β prevents the magnitudes of the elements of $\{\hat{Z}_n, n = 0,1,2,\ldots\}$ from becoming excessively small as k approaches zero. Knowledge of the elements of $\{\hat{Z}_n, n = 0,1,2,\ldots\}$ is equivalent to knowledge of the elements of $\{Z_n, n = 0, \pm 1, \pm 2,\ldots\}$ of (2).

The subroutine ZMAT(M1,M2,NP,NPHI,NT,RH,ZH,X,A,XT,AT,Z) is listed at the end of this section.. The subroutine ZMAT puts the elements of $\hat{Z}_n$ of (14) and (15) in Z((n-M1)*N*N+1) through Z((n-M1+1)*N*N) for n = M1, M1+1,M1+2,...M2 where M1 and M2 are non-negative integers and M2 $\geq$ M1. Here,

$$N = 2*NP - 3 \tag{17}$$

Storage of $\hat{Z}_n$ is by columns. Z is the only input argument. The rest of the arguments are input arguments and have the same meanings as in the subroutine ZMAT listed in [2, pp. 51-55]. NP is the number of data points on the generating curve of S. RH and ZH contain the coordinates of the data points in electrical length.

$$RH(J) = k\rho(t_J^-) \ , \quad J=1,2,\ldots NP \tag{18}$$

$$ZH(J) = kz(t_J^-) \ , \quad J=1,2,\ldots NP \tag{19}$$

Here, k is the propagation constant, z is the coordinate along the axis

about which the generating curve of S is rotated, and ρ is the distance from this axis. Also, (t_J^-) denotes evaluation at the Jth data point. NPHI is n_ϕ in the Gaussian quadrature formulas [2, Eqs. (64)-(66)]. X contains the n_ϕ abscissas $x_\ell^{(n_\phi)}$ of [2, Eq. (70)], and A contains the n_ϕ weights $A_\ell^{(n_\phi)}$ in [2, Eqs. (64)-(66)]. NT is n_t in the Gaussian quadrature formulas [2, Eqs. (62)-(63)]. XT contains the n_t abscissas $x_\ell^{(n_t)}$, in [2, Eqs. (62)-(63)], and AT contains the n_t weights $A_\ell^{(n_t)}$, in [2, Eqs. (62)-(63)]. The subroutine ZMAT calls the function BLOG which is listed in the next Section.

Minimum allocations in the subroutine ZMAT are given by

```
COMPLEX Z(M3*N*N), G4A(M3), G5A(M3), G6A(M3),
   G4B(M3), G5B(M3), G6B(M3), GA(NPHI), GB(NPHI)

DIMENSION RH(NP), ZH(NP), X(NPHI), A(NPHI),
   XT(NT), AT(NT), RS(NP-1), ZS(NP-1), D(NP-1),
   DR(NP-1), DZ(NP-1), DM(NP-1), C2(NPHI),
   C3(NPHI), R2(NT), Z2(NT), C4(M3*NPHI), C5(M3*NPHI),
   C6(M3*NPHI), Z7(NT), R7(NT), Z8(NT), R8(NT)
```

where N is given by (17) and

$$M3 = M2 - M1 + 1 \tag{20}$$

In view of (5), (14) becomes

$$\hat{Z}_0 = \begin{bmatrix} Z_0^{tt} & 0 \\ 0 & \beta Z_0^{\phi\phi} \end{bmatrix} \tag{21}$$

where the ijth elements of Z_0^{tt} and $Z_0^{\phi\phi}$ are given by [2, Eqs. (9) and (12)].

In [2, Eqs. (9) and (12)], the region of integration for which

$$\left. \begin{aligned} t_p^- \le t \le t_{p+1}^- \\ t_q^- \le t' \le t_{q+1}^- \end{aligned} \right\} \tag{22}$$

is called A_{pq}. The ranges of values of the subscripts p and q on A_{pq} are given by

$$\left. \begin{aligned} 1 \le p \le P-1 \\ 1 \le q \le P-1 \end{aligned} \right\} \tag{23}$$

where, as in [2], P is the number of data points on the generating curve of S. From [2, Eq. (48)], the contributions to the elements of Z_0^{tt} due to A_{pq} are

$$(\overset{*}{Z}{}_0^{tt})_{ij} = \frac{jk^2\Delta_p\Delta_q}{8}(G_{5a} \sin v_p \sin v_q + G_{7a} \cos v_p \cos v_q) +$$

$$\frac{(-1)^{q-j}\ jk^2\Delta_p\Delta_q}{8}(G_{5b} \sin v_p \sin v_q + G_{7b} \cos v_p \cos v_q) -$$

$$(-1)^{p+q-i-j}\ \frac{j}{2}\ G_{7a} \tag{24}$$

where i is either p-1 or p and j is either q-1 or q but neither i nor j can be 0 or P-1. The asterisk on the left-hand side of (24) denotes that $(\overset{*}{Z}{}_0^{tt})_{ij}$ is not all of $(Z_0^{tt})_{ij}$ but only the contribution due to A_{pq}. According to [2, Eq. (51)], the contribution to the elements of $\beta Z_0^{\phi\phi}$ due to A_{pq} is

$$\beta(Z_0^{\phi\phi})_{pq} = 2j\ \left(\frac{\beta k^2\Delta_p\Delta_q}{4}\right)\left(G_{5a} + \frac{\Delta_q \sin v_q}{2\rho_q}\ G_{5b}\right) \tag{25}$$

Equation (25) means that A_{pq} contributes only to $\beta(Z_0^{\phi\phi})_{pq}$ and gives all of $\beta(Z_0^{\phi\phi})_{pq}$.

Substitution of (2) into (15) gives

$$\hat{Z}_n = \begin{bmatrix} \beta Z_n^{mm} & \beta Z_n^{me} \\ \beta Z_n^{em} & \beta Z_n^{ee} \end{bmatrix} \quad , \; n=1,2,3,... \tag{26}$$

where the ijth elements of Z_n^{mm}, Z_n^{em}, Z_n^{me}, and Z_n^{ee} are given by (7), (8), (9), and 10, respectively. The contributions to the elements of βZ_n^{mm} due to A_{pq} are

$$\beta(\overset{*}{Z}{}_n^{mm})_{p-1,q-1} = \beta((\overset{*}{Z}{}_n^{tt})_{p-1,q-1} - \alpha_{nq}(\overset{*}{Z}{}_n^{t\phi})_{p-1,q} + \alpha_{np}(\overset{*}{Z}{}_n^{\phi t})_{p,q-1} - \alpha_{np}\alpha_{nq}(Z_n^{\phi\phi})_{pq}) \tag{27}$$

$$\beta(\overset{*}{Z}{}_n^{mm})_{p,q-1} = \beta((\overset{*}{Z}{}_n^{tt})_{p,q-1} - \alpha_{nq}(\overset{*}{Z}{}_n^{t\phi})_{pq} - \alpha_{np}(\overset{*}{Z}{}_n^{\phi t})_{p,q-1} + \alpha_{np}\alpha_{nq}(Z_n^{\phi\phi})_{pq}) \tag{28}$$

$$\beta(\overset{*}{Z}{}_n^{mm})_{p-1,q} = \beta((\overset{*}{Z}{}_n^{tt})_{p-1,q} + \alpha_{nq}(\overset{*}{Z}{}_n^{t\phi})_{p-1,q} + \alpha_{np}(\overset{*}{Z}{}_n^{\phi t})_{pq} + \alpha_{np}\alpha_{nq}(Z_n^{\phi\phi})_{pq}) \tag{29}$$

$$\beta(\overset{*}{Z}{}_n^{mm})_{pq} = \beta((\overset{*}{Z}{}_n^{tt})_{pq} + \alpha_{nq}(\overset{*}{Z}{}_n^{t\phi})_{pq} - \alpha_{np}(\overset{*}{Z}{}_n^{\phi t})_{pq} - \alpha_{np}\alpha_{nq}(Z_n^{\phi\phi})_{pq}) \tag{30}$$

The asterisks above the Z_n's in (27)-(30) denote the contributions due to A_{pq}. The contributions to the elements of βZ_n^{em} due to A_{pq} are

$$\beta(\overset{*}{Z}{}_n^{em})_{p,q-1} = \beta k\rho_p((\overset{*}{Z}{}_n^{\phi t})_{p,q-1} - \alpha_{nq}(Z_n^{\phi\phi})_{pq}) \tag{31}$$

$$\beta(\overset{*}{Z}{}_n^{em})_{pq} = \beta k\rho_p((\overset{*}{Z}{}_n^{\phi t})_{pq} + \alpha_{nq}(Z_n^{\phi\phi})_{pq}) \tag{32}$$

The contributions to the elements of βZ_n^{me} due to A_{pq} are

$$\beta(\overset{*}{Z}{}_n^{me})_{p-1,q} = \beta k\rho_q((\overset{*}{Z}{}_n^{t\phi})_{p-1,q} + \alpha_{np}(Z_n^{\phi\phi})_{pq}) \tag{33}$$

$$\beta(\overset{*}{Z}{}_n^{me})_{pq} = \beta k\rho_q((\overset{*}{Z}{}_n^{t\phi})_{pq} - \alpha_{np}(Z_n^{\phi\phi})_{pq}) \tag{34}$$

The asterisks above the Z_n's in (31)-(34) denote the contributions due to A_{pq}. As concerns Z_n^{ee}, A_{pq} contributes only to the pqth element of Z_n^{ee} and gives all of this element.

$$\beta(Z_n^{ee})_{pq} = \beta k^2 \rho_p \rho_q (Z_n^{\phi\phi})_{pq} \tag{35}$$

In (23), the subscripts p and q on A_{pq} run from 1 to P-1. However, i=1,2,...P-2 on the testing function $\underline{W}_{ni}^m$ of [1, Eq. (102)], and j=1,2,...P-2 on the expansion function $\underline{J}_{nj}^m$ of [1, Eq. (100)]. Therefore, some of (27)-(34) must be deleted when p is either 1 or P-1 or when q is either 1 or P-1. If p=1, then (27),(29), and (33) are to be deleted. If p = P-1, then (28), (30), and (34) are to be deleted. If q=1, then (27), (28), and (31) are to be deleted. If q = P-1, then (29), (30), and (32) are to be deleted.

The Z_n's on the right-hand sides of (27)-(35) are given by [2, Eqs. (19)-(22)]. If [2, Eqs. (19)-(22)] are substituted into (27)-(34), then, thanks to (11) and the formulas

$$\frac{d}{dt} T_p(t) = \frac{P_p(t)}{\Delta_p} \quad , \quad t_p^- < t \leq t_{p+1}^- \tag{36}$$

$$\frac{d}{dt} T_{p-1}(t) = -\frac{P_p(t)}{\Delta_p} \quad , \quad t_p^- < t \leq t_{p+1}^- \quad , \tag{37}$$

the last G_7 term in [2, Eq. (19)] and the G_7 terms in [2, Eqs. (20)-(22)] cancel each other. These terms are the electric scalar potential terms. The remaining terms in [2, Eqs. (19)-(22)] are the magnetic vector potential terms. Therefore, [2, Eqs. (19)-(22)] reduce (27)-(34) to

$$\beta(\overset{*}{Z}{}_n^{mm})_{p-1,q-1} = \beta((\overset{a}{Z}{}_n^{tt})_{p-1,q-1} - \alpha_{nq}(\overset{a}{Z}{}_n^{t\phi})_{p-1,q} + \alpha_{np}(\overset{a}{Z}{}_n^{\phi t})_{p,q-1} - \alpha_{np}\alpha_{nq}(\overset{a}{Z}{}_n^{\phi\phi})_{pq}) \tag{38}$$

$$\beta(\overset{*}{Z}{}_n^{mm})_{p,q-1} = \beta((\overset{a}{Z}{}_n^{tt})_{p,q-1} - \alpha_{nq}(\overset{a}{Z}{}_n^{t\phi})_{pq} - \alpha_{np}(\overset{a}{Z}{}_n^{\phi t})_{p,q-1} + \alpha_{np}\alpha_{nq}(\overset{a}{Z}{}_n^{\phi\phi})_{pq}) \tag{39}$$

$$\beta(\overset{*}{Z}{}_n^{mm})_{p-1,q} = \beta((\overset{a}{Z}{}_n^{tt})_{p-1,q} + \alpha_{nq}(\overset{a}{Z}{}_n^{t\phi})_{p-1,q} + \alpha_{np}(\overset{a}{Z}{}_n^{\phi t})_{pq} + \alpha_{np}\alpha_{nq}(\overset{a}{Z}{}_n^{\phi\phi})_{pq}) \tag{40}$$

$$\beta(\overset{*}{Z}{}_n^{mm})_{pq} = \beta((\overset{a}{Z}{}_n^{tt})_{pq} + \alpha_{nq}(\overset{a}{Z}{}_n^{t\phi})_{pq} - \alpha_{np}(\overset{a}{Z}{}_n^{\phi t})_{pq} - \alpha_{np}\alpha_{nq}(\overset{a}{Z}{}_n^{\phi\phi})_{pq}) \tag{41}$$

$$\beta(\overset{*}{Z}{}_n^{em})_{p,q-1} = \beta k\rho_p((\overset{a}{Z}{}_n^{\phi t})_{p,q-1} - \alpha_{nq}(\overset{a}{Z}{}_n^{\phi\phi})_{pq}) \tag{42}$$

$$\beta(\overset{*}{Z}{}_n^{em})_{pq} = \beta k\rho_p((\overset{a}{Z}{}_n^{\phi t})_{pq} + \alpha_{nq}(\overset{a}{Z}{}_n^{\phi\phi})_{pq}) \tag{43}$$

$$\beta(\overset{*}{Z}{}_n^{me})_{p-1,q} = \beta k\rho_q((\overset{a}{Z}{}_n^{t\phi})_{p-1,q} + \alpha_{np}(\overset{a}{Z}{}_n^{\phi\phi})_{pq}) \tag{44}$$

$$\beta(\overset{*}{Z}{}_n^{me})_{pq} = \beta k\rho_q((\overset{a}{Z}{}_n^{t\phi})_{pq} - \alpha_{np}(\overset{a}{Z}{}_n^{\phi\phi})_{pq}) \tag{45}$$

where the $\overset{a}{Z}_n$'s are the magnetic vector potential contributions to the Z_n's of [2, Eqs. (19)-(22)]. Equation (35) remains unchanged.

$$\beta(Z_n^{ee})_{pq} = \beta k^2\rho_p\rho_q(Z_n^{\phi\phi})_{pq} \tag{46}$$

As extracted from [2, Eqs. (19)-(22)], the Z_n's on the right-hand sides of (38)-(46) are

$$(\overset{a}{Z}{}_n^{tt})_{ij} = j\int_{\bar{t}_p}^{\bar{t}_{p+1}} dt \int_{\bar{t}_q}^{\bar{t}_{q+1}} dt'\; k^2 T_i(t)T_j(t')(G_5 \sin v \sin v' + G_7 \cos v \cos v') \tag{47}$$

$$(Z_n^{a\phi t})_{pj} = -\frac{1}{\rho_p}\int_{t_p^-}^{t_{p+1}^-} dt\ P_p(t)\int_{t_q^-}^{t_{q+1}^-} dt'(k^2\rho\ T_j(t')\ G_6\sin v') \tag{48}$$

$$(Z_n^{at\phi})_{iq} = \frac{1}{\rho_q}\int_{t_p^-}^{t_{p+1}^-} dt\int_{t_q^-}^{t_{q+1}^-} dt'\ P_q(t')(k^2\rho'\ T_i(t)\ G_6\sin v) \tag{49}$$

$$(Z_n^{a\phi\phi})_{pq} = \frac{j}{\rho_p\rho_q}\int_{t_p^-}^{t_{p+1}^-} dt\ P_p(t)\int_{t_q^-}^{t_{q+1}^-} dt'\ P_q(t')(k^2\rho\rho'\ G_5) \tag{50}$$

$$(Z_n^{\phi\phi})_{pq} = \frac{j}{\rho_p\rho_q}\int_{t_p^-}^{t_{p+1}^-} dt\ P_p(t)\int_{t_q^-}^{t_{q+1}^-} dt'\ P_q(t')(k^2\rho\rho'\ G_5 - n^2G_7) \tag{51}$$

where i is either p-1 or p and j is either q-1 or q, but neither i nor j can be 0 or P-1.

If the approximations that led from [2, Eqs. (19)-(22)] to [2, Eqs. (48)-(51)] are applied to (47)-(51), then (47)-(50) reduce to the vector potential terms in [2, Eqs. (48)-(51)] and (51) reduces to [2, Eq. (51)]. Hence,

$$(Z_n^{att})_{ij} = \frac{jk^2\Delta_p\Delta_q}{8}(G_{5a}\sin v_p\sin v_q + G_{7a}\cos v_p\cos v_q) +$$

$$\frac{(-1)^{q-j}jk^2\Delta_p\Delta_q}{8}(G_{5b}\sin v_p\sin v_q + G_{7b}\cos v_p\cos v_q) \tag{52}$$

$$(Z_n^{a\phi t})_{pj} = -\left(\frac{k^2\Delta_p\Delta_q\sin v_q}{4}\right)(G_{6a} + (-1)^{q-j}G_{6b}) \tag{53}$$

$$(Z_n^{at\phi})_{iq} = \left(\frac{k^2\Delta_p\Delta_q\sin v_p}{4}\right)\left(G_{6a} + \frac{\Delta_q\sin v_q}{2\rho_q}G_{6b}\right) \tag{54}$$

$$(\hat{Z}_n^{\phi\phi})_{pq} = 2j\left(\frac{k^2\Delta_p\Delta_q}{4}\right)\left(G_{5a} + \frac{\Delta_q \sin v_q}{2\rho_q} G_{5b}\right) \qquad (55)$$

$$(Z_n^{\phi\phi})_{pq} = (\hat{Z}_n^{\phi\phi})_{pq} - 2j\left(\frac{n\Delta_q}{2\rho_q}\right)\left(\frac{n\Delta_p}{2\rho_p}\right) G_{7a} \qquad (56)$$

where i is either p-1 or p and j is either q-1 or q, but neither i nor j can be 0 or P-1.

The developments that have occurred from (21) to (56) are summarized as follows. It was shown that $\hat{Z}_0$ is given by (21) where the contributions to the elements of Z_0^{tt} due to A_{pq} are given by (24) and the elements of $\beta Z_0^{\phi\phi}$ are given by (25). Equation (26) was established for $\hat{Z}_n$. The contributions to the matrix elements on the right-hand side of (26) due to A_{pq} are given by (38)-(46). The Z_n's on the right-hand sides of (38)-(46) are given by (52)-(56).

The subroutine ZMAT is similar to the subroutine described and listed in [2, pp. 43-55]. Hence, it suffices to point out statements that differ from those in the subroutine listed in [2, pp. 51-55]. In the listing of ZMAT at the end of this Section, line 53 puts β of (16) in RD.

The index JQ of DO loop 15 obtains the subscript q on A_{pq} of (22). This q appears on the right-hand sides of (24)-(25), (38)-(46), and (52)-(56). With reference to (44) and (45), line 59 puts $\beta k\rho_q$ in RQ. The index IP of DO loop 16 obtains the subscript p which appears on the right-hand sides of (24)-(25), (38)-(46), and (52)-(56). With reference to (42) and (43), line 81 puts $\beta k\rho_p$ in RP. Line 260 puts $\beta k^2\rho_p\rho_q$ of (46) in RPQ.

Inside DO loop 31, the elements of $\hat{Z}_n$ are obtained for

$$n = M1-1 + M \qquad (57)$$

where M is the index of DO loop 31. Table 1 describes the action of some statements in DO loop 31. The statement whose line number is given in the third column of Table 1 stores the text quantity of the second column in the variable in ZMAT listed in the first column. Integers {KM, M=1,2,...8} are defined by lines 272-279. Lines 317-343 set Z(KM) equal to VM for M=4,6, and 8, and add VM to Z(KM) for M=1,2,3,5, and 7. The branch statements among lines 317-343 prevent alteration of Z(KM) at the forbidden values of p and q in [2, Table 2 on p. 50].

Table 1. Variables in ZMAT versus text quantities

Variable in ZMAT	Text Quantity	Line Number
FM	n	262
H5A	G_{5a}	263
H5B	G_{5b}	264
H4A	G_{7a}	265
H4B	G_{7b}	266
U7	$\frac{j}{8}\, k^2 \Delta_p \Delta_q (G_{5a} \sin v_p \sin v_q + G_{7a} \cos v_p \cos v_q)$	267
U8	$\frac{j}{8}\, k^2 \Delta_p \Delta_q (G_{5b} \sin v_p \sin v_q + G_{7b} \cos v_p \cos v_q)$	268
U5	$(\overset{a}{Z}{}_n^{tt})_{i,q-1}$ of (52)	269
U6	$(\overset{a}{Z}{}_n^{tt})_{iq}$ of (52)	270
V9	$(\overset{a}{Z}{}_n^{\phi\phi})_{pq}$ of (55)	271
U7	$\frac{-j}{2}\, G_{7a}$	281
V1	$(\overset{*}{Z}{}_0^{tt})_{p-1,q-1}$ of (24)	282
V2	$(\overset{*}{Z}{}_0^{tt})_{p,q-1}$ of (24)	283
V3	$(\overset{*}{Z}{}_0^{tt})_{p-1,q}$ of (24)	284
V4	$(\overset{*}{Z}{}_0^{tt})_{pq}$ of (24)	285
Z(K8+MT)	$\beta(Z_0^{\phi\phi})_{pq}$ of (25)	290
H6A	G_{6a}	292
H6B	G_{6b}	293

Continuation of Table 1

U7	$(\overset{a\phi t}{Z_n})_{p,q-1}$ of (53)	294
U8	$(\overset{a\phi t}{Z_n})_{pq}$ of (53)	295
UC	$(\overset{at\phi}{Z_n})_{iq}$ of (54)	296
A1	$\frac{n\Delta_p}{2\rho_p}$	297
FMD	$\frac{n\Delta_q}{2\rho_q}$	298
AP	α_{np}	299
AQ	α_{nq}	300
UD	$\alpha_{nq}\ (\overset{a\phi\phi}{Z_n})_{pq}$	301
V5	$(\overset{a\phi t}{Z_n})_{p,q-1} - \alpha_{nq}\ (\overset{a\phi\phi}{Z_n})_{pq}$	302
V6	$(\overset{a\phi t}{Z_n})_{pq} + \alpha_{nq}\ (\overset{a\phi\phi}{Z_n})_{pq}$	303
V7	$\alpha_{np}\ (\overset{a\phi t}{Z_n})_{p,q-1} - \alpha_{np}\alpha_{nq}\ (\overset{a\phi\phi}{Z_n})_{pq}$	304
V8	$\alpha_{np}\ (\overset{a\phi t}{Z_n})_{pq} + \alpha_{np}\alpha_{nq}\ (\overset{a\phi\phi}{Z_n})_{pq}$	305
UD	$\alpha_{nq}\ (\overset{at\phi}{Z_n})_{iq}$	306
V1	$\beta(\overset{*mm}{Z_n})_{p-1,q-1}$ of (38)	307
V2	$\beta(\overset{*mm}{Z_n})_{p,q-1}$ of (39)	308
V3	$\beta(\overset{*mm}{Z_n})_{p-1,q}$ of (40)	309
V4	$\beta(\overset{*mm}{Z_n})_{pq}$ of (41)	310

Continuation of Table 1

V5	$\beta(\overset{*}{Z}{}_n^{em})_{p,q-1}$ of (42)	311
V6	$\beta(\overset{*}{Z}{}_n^{em})_{pq}$ of (43)	312
UD	$\alpha_{np}\ (Z_n^{a\phi\phi})_{pq}$	313
V7	$\beta(\overset{*}{Z}{}_n^{me})_{p-1,q}$ of (44)	314
V8	$\beta(\overset{*}{Z}{}_n^{me})_{pq}$ of (45)	315
Z(K8+MT)	$\beta(Z_n^{ee})_{pq}$ of (46)	316

```
C     LISTING OF THE SUBROUTINE ZMAT
C     THE SUBROUTINE ZMAT CALLS THE FUNCTION BLOG
      SUBROUTINE ZMAT(M1,M2,NP,NPHI,NT,RH,ZH,X,A,XT,AT,Z)
      COMPLEX Z(1600),U1,U2,U3,U4,U5,U6,U7,U8,UA,UB,G4A(10),G5A(10)
      COMPLEX CMPLX,G6A(10),G4B(10),G5B(10),G6B(10),H4A,H5A,H6A,H4B,H5B
      COMPLEX H6B,UC,UD,GA(48),GB(48),AP,AQ,V1,V2,V3,V4,V5,V6,V7,V8,V9
      DIMENSION RH(43),ZH(43),X(48),A(48),XT(10),AT(10),RS(42),ZS(42)
      DIMENSION D(42),DR(42),DZ(42),DM(42),C2(48),C3(48),R2(10),Z2(10)
      DIMENSION C4(200),C5(200),C6(200),Z7(10),R7(10),Z8(10),R8(10)
      CT=2.
      CP=.1
      DO 10 I=2,NP
      I2=I-1
      RS(I2)=.5*(RH(I)+RH(I2))
      ZS(I2)=.5*(ZH(I)+ZH(I2))
      D1=.5*(RH(I)-RH(I2))
      D2=.5*(ZH(I)-ZH(I2))
      D(I2)=SQRT(D1*D1+D2*D2)
      DR(I2)=D1
      DZ(I2)=D2
      DM(I2)=D(I2)/RS(I2)
   10 CONTINUE
      M3=M2-M1+1
      M4=M1-1
      PI2=1.570796
      DO 11 K=1,NPHI
      PH=PI2*(X(K)+1.)
      C2(K)=PH*PH
      SN=SIN(.5*PH)
      C3(K)=4.*SN*SN
      A1=PI2*A(K)
      D4=.5*A1*C3(K)
      D5=A1*COS(PH)
      D6=A1*SIN(PH)
      M5=K
      DO 29 M=1,M3
      PHM=(M4+M)*PH
      A2=COS(PHM)
      C4(M5)=D4*A2
      C5(M5)=D5*A2
      C6(M5)=D6*SIN(PHM)
      M5=M5+NPHI
   29 CONTINUE
   11 CONTINUE
      MP=NP-1
      MT=MP-1
      N=MT+MP
      N2N=MT*N
      N2=N*N
      U1=(0.,.5)
      U2=(0.,2.)
      JN=-1-N
      RD=1./(D(1)*D(1))
      DO 15 JQ=1,MP
      KQ=2
      IF(JQ.EQ.1) KQ=1
      IF(JQ.EQ.MP) KQ=3
      R1=RS(JQ)
      RQ=R1*RD
      Z1=ZS(JQ)
```

```
      D1=D(JQ)
      D2=DR(JQ)
      D3=DZ(JC)
      D4=D2/R1
      D5=DM(JQ)
      SV=D2/D1
      CV=D3/D1
      T6=CT*D1
      T62=T6+D1
      T62=T62*T62
      R6=CP*R1
      R62=R6*R6
      DC 12 L=1,NT
      R2(L)=R1+D2*XT(L)
      Z2(L)=Z1+D3*XT(L)
   12 CCNTINUE
      U3=D2*U1
      U4=D3*U1
      DO 16 IP=1,MP
      R3=RS(IP)
      RP=R3*RD
      Z3=ZS(IP)
      R4=R1-R3
      Z4=Z1-Z3
      FM=R4*SV+Z4*CV
      PHM=ABS(FM)
      PH=ABS(R4*CV-Z4*SV)
      D6=PH
      IF(PHM.LE.D1) GO TO 26
      D6=PHM-D1
      D6=SQRT(D6*D6+PH*PH)
   26 IF(IP.EQ.JQ) GC TC 27
      KP=1
      IF(T6.GT.D6) KP=2
      IF(R6.GT.D6) KP=3
      GO TO 28
   27 KP=4
   28 GC TO (41,42,41,42),KP
   42 DO 40 L=1,NT
      D7=R2(L)-R3
      D8=Z2(L)-Z3
      Z7(L)=D7*D7+D8*D8
      R7(L)=R3*R2(L)
      Z8(L)=.25*Z7(L)
      R8(L)=.25*R7(L)
   40 CONTINUE
      Z4=R4*R4+Z4*Z4
      R4=R3*R1
      R5=.5*R3*SV
      DO 33 K=1,NPHI
      A1=C3(K)
      RR=Z4+R4*A1
      UA=0.
      UB=0.
      IF(RR.LT.T62) GO TO 34
      DO 35 L=1,NT
      R=SCRT(Z7(L)+R7(L)*A1)
      SN=-SIN(R)
      CS=COS(R)
      UC=AT(L)/R*CMPLX(CS,SN)
```

```
      UA=UA+UC
      UB=XT(L)*UC+UB
   35 CCNTINUE
      GO TO 36
   34 DO 37 L=1,NT
      R=SCRT(Z8(L)+R8(L)*A1)
      SN=-SIN(R)
      CS=COS(R)
      UC=AT(L)/R*SN*CMPLX(-SN,CS)
      UA=UA+UC
      UB=XT(L)*UC+UB
   37 CONTINUE
      A2=FM+R5*A1
      D9=RR-A2*A2
      R=ABS(A2)
      D7=R-D1
      D8=R+D1
      D6=SQRT(D8*D8+D9)
      R=SCRT(D7*D7+D9)
      IF(D7.GE.0.) GO TO 38
      A1=ALCG((D8+D6)*(-D7+R)/D9)/D1
      GO TO 39
   38 A1=ALOG((D8+D6)/(D7+R))/D1
   39 UA=A1+UA
      UB=A2*(4./(D6+R)-A1)/D1+UB
   36 GA(K)=UA
      GB(K)=UB
   33 CONTINUE
      K1=0
      DO 45 M=1,M3
      H4A=0.
      H5A=0.
      H6A=0.
      H4B=0.
      H5B=0.
      H6B=0.
      DO 46 K=1,NPHI
      K1=K1+1
      D6=C4(K1)
      D7=C5(K1)
      D8=C6(K1)
      UA=GA(K)
      UB=GB(K)
      H4A=D6*UA+H4A
      H5A=D7*UA+H5A
      H6A=D8*UA+H6A
      H4B=D6*UB+H4B
      H5B=D7*UB+H5B
      H6B=D8*UB+H6B
   46 CONTINUE
      G4A(M)=H4A
      G5A(M)=H5A
      G6A(M)=H6A
      G4B(M)=H4B
      G5B(M)=H5B
      G6B(M)=H6B
   45 CCNTINUE
      IF(KP.NE.4) GO TO 47
      A2=D1/(PI2*R1)
      D6=2./D1
```

```
      D8=0.
      DO 63 K=1.NPHI
      A1=R4*C2(K)
      R=R4*C3(K)
      IF(R.LT.T62) GO TO 64
      D7=0.
      DO 65 L=1.NT
      D7=D7+AT(L)/SQRT(Z7(L)+A1)
   65 CONTINUE
      GO TO 66
   64 A1=A2/(X(K)+1.)
      D7=D6*ALOG(A1+SQRT(1.+A1*A1))
   66 D8=D8+A(K)*D7
   63 CONTINUE
      A1=.5*A2
      A2=1./A1
      D8=-PI2*D8+2./R1*(BLOG(A2)+A2*BLOG(A1))
      DO 67 M=1.M3
      G5A(M)=D8+G5A(M)
   67 CONTINUE
      GO TO 47
   41 DO 25 M=1.M3
      G4A(M)=0.
      G5A(M)=0.
      G6A(M)=0.
      G4B(M)=0.
      G5B(M)=0.
      G6B(M)=0.
   25 CONTINUE
      DO 13 L=1.NT
      A1=R2(L)
      R4=A1-R3
      Z4=Z2(L)-Z3
      Z4=R4*R4+Z4*Z4
      R4=R3*A1
      DO 17 K=1.NPHI
      R=SQRT(Z4+R4*C3(K))
      SN=-SIN(R)
      CS=COS(R)
      GA(K)=CMPLX(CS,SN)/R
   17 CONTINUE
      D6=0.
      IF(R62.LE.Z4) GO TO 51
      DO 62 K=1.NPHI
      D6=D6+A(K)/SQRT(Z4+R4*C2(K))
   62 CONTINUE
      Z4=3.141593/SQRT(Z4/R4)
      D6=-PI2*D6+ALOG(Z4+SQRT(1.+Z4*Z4))/SQRT(R4)
   51 A1=AT(L)
      A2=XT(L)*A1
      K1=0
      DO 30 M=1.M3
      U5=0.
      U6=0.
      U7=0.
      DO 32 K=1.NPHI
      UA=GA(K)
      K1=K1+1
      U5=C4(K1)*UA+U5
      U6=C5(K1)*UA+U6
```

```
      U7=C6(K1)*UA+U7
   32 CONTINUE
      U6=D6+U6
      G4A(M)=A1*U5+G4A(M)
      G5A(M)=A1*U6+G5A(M)
      G6A(M)=A1*U7+G6A(M)
      G4B(M)=A2*U5+G4B(M)
      G5B(M)=A2*U6+G5B(M)
      G6B(M)=A2*U7+G6B(M)
   30 CONTINUE
   13 CONTINUE
   47 A1=DR(IP)
      UA=A1*U3
      UB=DZ(IP)*U4
      A2=D(IP)
      D6=-A2*D2
      D7=D1*A1
      D8=D1*A2
      JM=JN
      RPQ=R1*RP
      DO 31 M=1,M3
      FM=M4+M
      H5A=G5A(M)
      H5B=G5B(M)
      H4A=G4A(M)+H5A
      H4B=G4B(M)+H5B
      U7=UA*H5A+UB*H4A
      U8=UA*H5B+UB*H4B
      U5=U7-U8
      U6=U7+U8
      V9=U2*D8*(H5A+D4*H5B)
      K1=IP+JM
      K2=K1+1
      K3=K1+N
      K4=K2+N
      K5=K2+MT
      K6=K4+MT
      K7=K3+N2N
      K8=K4+N2N
      IF(FM.NE.0.) GO TO 14
      U7=-U1*H4A
      V1=U5+U7
      V2=U5-U7
      V3=U6-U7
      V4=U6+U7
      V5=0.
      V6=0.
      V7=0.
      V8=0.
      Z(K8+MT)=RD*V9
      GO TO 43
   14 H6A=G6A(M)
      H6B=G6B(M)
      U7=D6*(H6A-H6B)
      U8=D6*(H6A+H6B)
      U9=D7*(H6A+D4*H6B)
      A1=FM*DM(IP)
      FMD=FM*D5
      AP=1./A1*U1
      AG=(1./FMD)*U1
```

```
      UD=AQ*V9
      V5=U7-UD
      V6=U8+UD
      V7=AP*V5
      V8=AP*V6
      UD=AQ*UC
      V1=RD*(U5-UD+V7)
      V2=RD*(U5-UD-V7)
      V3=RD*(U6+UD+V8)
      V4=RD*(U6+UD-V8)
      V5=RP*V5
      V6=RP*V6
      UD=AP*V9
      V7=RQ*(UC+UD)
      V8=RQ*(UC-UD)
      Z(K8+MT)=RPQ*(V9-A1*FMD*H4A*U2)
      GO TO (18,20,19),KQ
      Z(K6)=V6
      IF(IP.EQ.1) GO TO 21
      Z(K3)=Z(K3)+V3
      Z(K7)=Z(K7)+V7
      IF(IP.EQ.MP) GO TO 22
      Z(K4)=V4
      Z(K8)=V8
      GO TO 22
      Z(K5)=Z(K5)+V5
      IF(IP.EQ.1) GO TO 23
      Z(K1)=Z(K1)+V1
      Z(K7)=Z(K7)+V7
      IF(IP.EQ.MP) GO TO 22
      Z(K2)=Z(K2)+V2
      Z(K8)=V8
      GO TO 22
      Z(K5)=Z(K5)+V5
      Z(K6)=V6
      IF(IP.EQ.1) GO TO 24
      Z(K1)=Z(K1)+V1
      Z(K3)=Z(K3)+V3
      Z(K7)=Z(K7)+V7
      IF(IP.EQ.MP) GO TO 22
      Z(K2)=Z(K2)+V2
      Z(K4)=V4
      Z(K8)=V8
      JM=JM+N2
      CONTINUE
      CONTINUE
      JN=JN+N
      CONTINUE
      RETURN
      END
```

III. THE FUNCTION BLOG

The function BLOG(x) calculates $\log(x + \sqrt{1 + x^2})$ for $x \geq 0$. The method of calculation is described in [2, p. 56].

```
C     LISTING OF THE FUNCTION BLOG
      FUNCTION BLOG(X)
      IF(X.GT..1) GO TO 1
      X2=X*X
      BLOG=((.075*X2-.1666667)*X2+1.)*X
      RETURN
    1 BLOG=ALOG(X+SQRT(1.+X*X))
      RETURN
      END
```

IV. THE SUBROUTINE PLANE

The subroutine PLANE calculates the elements of $\vec{V}_n$ of (4). According to [1, Eqs. (61) and (62)], the ith elements of the column vectors $\vec{V}^m_n$ and $\vec{V}^e_n$ on the right-hand side of (4) are given by

$$V^m_{ni} = \frac{1}{\eta} \iint_S \underline{W}^m_{ni} \cdot \underline{E}^{inc} ds \tag{58}$$

$$V^e_{ni} = \frac{k\rho_i}{\eta} \iint_S \underline{W}^e_{ni} \cdot \underline{E}^{inc} ds \tag{59}$$

Expressions (58) and (59) are calculated for $\underline{E}^{inc}$ equal to the θ-polarized plane wave electric field $\underline{E}^\theta$ given by

$$\underline{E}^\theta = \underline{u}^t_\theta \, k\eta \, e^{-j\underline{k}_t \cdot \underline{r}} \tag{60}$$

and also for $\underline{E}^{inc}$ equal to the ϕ-polarized plane wave electric field $\underline{E}^\phi$ given by

$$\underline{E}^\phi = \underline{u}^t_\phi \, k\eta \, e^{-j\underline{k}_t \cdot \underline{r}} \tag{61}$$

In (60) and (61),

$$\underline{k}_t = - k(\underline{u}_x \sin \theta_t + \underline{u}_z \cos \theta_t) \tag{62}$$

$$\underline{u}^t_\theta = \underline{u}_x \cos \theta_t - \underline{u}_z \sin \theta_t \tag{63}$$

$$\underline{u}^t_\phi = \underline{u}_y \tag{64}$$

where θ_t is the angle of incidence and where $\underline{u}_x$, $\underline{u}_y$ and $\underline{u}_z$ are unit vectors in the x,y, and z directions, respectively. Also, $\underline{r}$ is the radius vector from the origin. The origin must lie on the axis about which the generating

curve of S is rotated because this axis is the z axis. The electric fields (60) and (61) are the same as [2, Eqs. (108) and (109)].

If $\vec{V}_n^m$ due to $\underline{E}^\theta$ is called $\vec{V}_n^{m\theta}$ and if $\vec{V}_n^e$ due to $\underline{E}^\theta$ is called $\vec{V}_n^{e\theta}$, then, according to (58) and (59), the ith elements of $\vec{V}_n^{m\theta}$ and $\vec{V}_n^{e\theta}$ are given by

$$V_{ni}^{m\theta} = \frac{1}{\eta} \iint_S \underline{W}_{ni}^m \cdot \underline{E}^\theta ds \tag{65}$$

$$V_{ni}^{e\theta} = \frac{k\rho_i}{\eta} \iint_S \underline{W}_{ni}^e \cdot \underline{E}^\theta ds \tag{66}$$

If $\vec{V}_n^m$ due to $\underline{E}^\phi$ is called $\vec{V}_n^{m\phi}$ and if $\vec{V}_n^e$ due to $\underline{E}^\phi$ is called $\vec{V}_n^{e\phi}$, then the ith elements of $\vec{V}_n^{m\phi}$ and $\vec{V}_n^{e\phi}$ are given by

$$V_{ni}^{m\phi} = \frac{1}{\eta} \iint_S \underline{W}_{ni}^m \cdot \underline{E}^\phi ds \tag{67}$$

$$V_{ni}^{e\phi} = \frac{k\rho_i}{\eta} \iint_S \underline{W}_{ni}^e \cdot \underline{E}^\phi ds \tag{68}$$

For n=0, the testing functions $\underline{W}_{0i}^m$ and $k\rho_i\underline{W}_{0i}^e$ are given by [1, Eqs. (93) and (94)]. Hence,

$$\underline{W}_{0i}^m = \underline{W}_{0i}^\phi \tag{69}$$

$$k\rho_i\underline{W}_{0i}^e = \underline{W}_{0i}^t \tag{70}$$

where $\underline{W}_{0i}^t$ and $\underline{W}_{0i}^\phi$ are defined by [2, Eqs. (4) and (5)]. Thanks to (69) and (70), each V_{0i} defined by (65)-(68) is equal to one of the V_{0i}'s defined by [2, Eqs. (113), (114), (116), and (117)]. More precisely,

$$\left.\begin{aligned} V^{m\theta}_{0i} &= V^{\phi\theta}_{0i} \\ V^{e\theta}_{0i} &= V^{t\theta}_{0i} \\ V^{m\phi}_{0i} &= V^{\phi\phi}_{0i} \\ V^{e\phi}_{0i} &= V^{t\phi}_{0i} \end{aligned}\right. \tag{71}$$

where the V's on the right-hand side of (71) are given by [2, Eqs. (113), (114), (116), and (117)]. It is evident from [2, Eqs. (114) and (116)] that

$$\left.\begin{aligned} V^{\phi\theta}_{0i} &= 0 \\ V^{t\phi}_{0i} &= 0 \end{aligned}\right\} \tag{72}$$

As a result of (71) and (72),

$$\left.\begin{aligned} \vec{V}^{m\theta}_{0} &= 0 \\ \vec{V}^{e\phi}_{0} &= 0 \end{aligned}\right\} \tag{73}$$

For $n \neq 0$, $\underline{W}^{m}_{ni}$ and $k\rho_i \underline{W}^{e}_{ni}$ are given by [1, Eqs. (102) and (103)]. Thanks to [1, Eq. (103)], (66) and (68) become

$$V^{e\theta}_{ni} = k\rho_i V^{\phi\theta}_{ni} \tag{74}$$

$$V^{e\phi}_{ni} = k\rho_i V^{\phi\phi}_{ni} \tag{75}$$

where $V^{\phi\theta}_{ni}$ and $V^{\phi\phi}_{ni}$ are given by [2, Eqs. (114) and (117)]. Instead of (65) and (67), [1, Eq. (68)] is used to obtain $V^{m\theta}_{ni}$ and $V^{m\phi}_{ni}$. The incident magnetic field $\underline{H}^{inc}$ appears in [1, Eq. (68)]. The incident magnetic field associated with the θ-polarized electric field $\underline{E}^{\theta}$ of (60) is called $\underline{H}^{\theta}$ and is given by

$$\underline{H}^{\theta} = - \underline{u}_{\phi}^{t} \, k \, e^{-j\underline{k}_t \cdot \underline{r}} \tag{76}$$

The incident magnetic field associated with the ϕ-polarized electric field $\underline{E}^{\phi}$ of (61) is called $\underline{H}^{\phi}$ and is given by

$$\underline{H}^{\phi} = \underline{u}_{\theta}^{t} \, k \, e^{-j\underline{k}_t \cdot \underline{r}} \tag{77}$$

Substitution of [1, Eq. (105)] for u_i and $\underline{H}^{\theta}$ for $\underline{H}^{inc}$ in [1, Eq. (68)] gives

$$V_{ni}^{m\theta} = - \frac{k}{n} \iint\limits_{S} T_i(t) \, e^{-jn\phi}(\underline{H}^{\theta} \cdot \underline{n})ds \tag{78}$$

Substitution of [1, Eq. (105)] for u_i and $\underline{H}^{\phi}$ for $\underline{H}^{inc}$ in [1, Eq. (68)] gives

$$V_{ni}^{m\phi} = - \frac{k}{n} \iint\limits_{S} T_i(t) \, e^{-jn\phi}(\underline{H}^{\phi} \cdot \underline{n})ds \tag{79}$$

If ds is replaced by $\rho d\phi \, dt$, (78) and (79) become

$$V_{ni}^{m\theta} = - \frac{k}{n} \int_{t_i^-}^{t_{i+2}^-} dt \, \rho \, T_i(t) \int_{-\pi}^{\pi} d\phi(\underline{H}^{\theta} \cdot \underline{n})e^{-jn\phi} \tag{80}$$

$$V_{ni}^{m\phi} = - \frac{k}{n} \int_{t_i^-}^{t_{i+2}^-} dt \, \rho \, T_i(t) \int_{-\pi}^{\pi} d\phi(\underline{H}^{\phi} \cdot \underline{n})e^{-jn\phi} \tag{81}$$

The radius vector $\underline{r}$ that appears in expressions (76) and (77) for $\underline{H}^{\theta}$ and $\underline{H}^{\phi}$ is given by

$$\underline{r} = \underline{u}_x \rho \cos \phi + \underline{u}_y \rho \sin \phi + \underline{u}_z z \tag{82}$$

From (62) and (82), we obtain

$$-j\underline{k}_t \cdot \underline{r} = jk(\rho \sin \theta_t \cos \phi + z \cos \theta_t) \tag{83}$$

The unit normal vector $\underline{n}$ that appears in (80) and (81) is given by

$$\underline{n} = \underline{u}_x \cos v \cos \phi + \underline{u}_y \cos v \sin \phi - \underline{u}_z \sin v \tag{84}$$

Equations (76), (64), (84), and (83) lead to

$$\underline{H}^{\theta} \cdot \underline{n} = - k \cos v \sin \phi \, e^{jk(\rho \sin \theta_t \cos \phi + z \cos \theta_t)} \tag{85}$$

Similarly, (77), (63), (84), and (83) lead to

$$\underline{H}^{\phi} \cdot \underline{n} = k(\cos \theta_t \cos v \cos \phi + \sin \theta_t \sin v) e^{jk(\rho \sin \theta_t \cos \phi + z \cos \theta_t)} \tag{86}$$

Now, $\underline{H}^{\theta} \cdot \underline{n}$ of (85) is odd in ϕ so that $V_{ni}^{m\theta}$ of (80) is even in n. Moreover, $\underline{H}^{\phi} \cdot \underline{n}$ of (86) is even in ϕ so that $V_{ni}^{m\phi}$ of (81) is odd in n. Because $V_{ni}^{\phi\theta}$ of [2, Eq. (114)] is odd in n, $V_{ni}^{e\theta}$ of (74) is odd in n. Because $V_{ni}^{\phi\phi}$ of [2, Eq. (117)] is even in n, $V_{ni}^{e\phi}$ of (75) is even in n. Hence,

$$\begin{bmatrix} \vec{V}_{-n}^{m\theta} & \vec{V}_{-n}^{m\phi} \\ \vec{V}_{-n}^{e\theta} & \vec{V}_{-n}^{e\phi} \end{bmatrix} = \begin{bmatrix} \vec{V}_{n}^{m\theta} & -\vec{V}_{n}^{m\phi} \\ -\vec{V}_{n}^{e\theta} & \vec{V}_{n}^{e\phi} \end{bmatrix}, \quad n=1,2,... \tag{87}$$

Therefore, it suffices to calculate the elements of the $\vec{V}_n$'s in (87) only for positive values of n.

Matrices $\hat{V}_n$ are defined by

$$\hat{V}_0 = \begin{bmatrix} \vec{V}_0^{e\theta} & 0 \\ 0 & \beta \vec{V}_0^{m\phi} \end{bmatrix} \tag{88}$$

$$\hat{V}_n = \beta \begin{bmatrix} \vec{V}_n^{m\theta} & \vec{V}_n^{m\phi} \\ \vec{V}_n^{e\theta} & \vec{V}_n^{e\phi} \end{bmatrix}, \quad n = \pm 1, \pm 2, ... \tag{89}$$

where β is given by (16). In view of (6) and (73), the matrices $\hat{Z}_n$ of (14)-(15) and $\hat{V}_n$ of (88)-(89) allow (1) to be rewritten as

$$\hat{Z}_n \hat{I}_n = \hat{V}_n \quad , \quad n = 0, \pm 1, \pm 2, \ldots \tag{90}$$

where

$$\hat{I}_0 = \begin{bmatrix} \vec{I}_0^{e\theta} & 0 \\ 0 & \vec{I}_o^{m\phi} \end{bmatrix} \tag{91}$$

$$\hat{I}_n = \begin{bmatrix} \vec{I}_n^{m\theta} & \vec{I}_n^{m\phi} \\ \vec{I}_n^{e\theta} & \vec{I}_n^{e\phi} \end{bmatrix} , \quad n = \pm 1, \pm 2, \ldots \tag{92}$$

In (91) and (92), $\vec{I}_n^{m\theta}$ is $\vec{I}_n^{m}$ due to $(\underline{E}^{\theta},\underline{H}^{\theta})$, $\vec{I}_n^{e\theta}$ is $\vec{I}_n^{e}$ due to $(\underline{E}^{\theta},\underline{H}^{\theta})$, $\vec{I}_n^{m\phi}$ is $\vec{I}_n^{m}$ due to $(\underline{E}^{\phi},\underline{H}^{\phi})$, and $\vec{I}_n^{e\phi}$ is $\vec{I}_n^{e}$ due to $(\underline{E}^{\phi},\underline{H}^{\phi})$.

The subroutine PLANE (M1,M2,NF,NP,NT,RH,ZH,XT,AT,THR,R) is listed at the end of this section. The subroutine PLANE puts the elements of $\hat{V}_{-n}$ of (88) and (89) in R((K-1)*M3*2*N + (n-M1)*2*N+1) through R((K-1)*M3*2*N + (n-M1+1)*2*N) for

$$\theta_t = \mathrm{THR(K)} \tag{93}$$

Here

$$\mathrm{N = 2*NP-3} \tag{94}$$

and

$$\mathrm{M3 = M2 - M1+1} \tag{95}$$

Also, n = M1, M1+1, M1+2...M2 where M1 and M2 are non-negative integers and M2 $\geq$ M1. Furthermore, K = 1,2,...NF. Storage of $\hat{V}_{-n}$ in R is by columns. R is the only output argument. The rest of the arguments are

input arguments and have the same meanings as in the subroutine PLANE listed in [2, pp. 61-62]. NP is the number of data points on the generating curve of S. RH and ZH are defined by (18) and (19). NT is n_T in the Gaussian quadrature formulas [2, Eqs. (132) and (133)]. XT contains the n_T abscissas $x_\ell^{(n_T)}$ in [2, Eqs. (132)-(135)], and AT contains the n_T weights $A_\ell^{(n_T)}$ in [2, Eqs. (132) and (133)]. THR is in radians.

Minimum allocations in the subroutine PLANE are given by

```
COMPLEX R(NF*M3*2*N), FA(M2+3), FB(M2+3), FC(M2+3)
DIMENSION RH(NP), ZH(NP), XT(NT), AT(NT), THR(NF),
     CS(NF), SN(NF), R2(NT), Z2(NT)
```

where N and M3 are given by (94) and (95), respectively.

Thanks to (71), the ith elements of the column vectors $\vec{V}_0^{e\theta}$ and $\beta V_0^{m\phi}$ in (88) are given by

$$V_{0i}^{e\theta} = V_{0i}^{t\theta} \tag{96}$$

$$\beta V_{0i}^{m\phi} = \beta V_{0i}^{\phi\phi} \tag{97}$$

where $V_{0i}^{t\theta}$ and $V_{0i}^{\phi\phi}$ are given by [2, Eqs. (113) and (117)]. If the contribution to $V_{0i}^{e\theta}$ of (96) due to the integration from t_p^- to t_{p+1}^- is called $\overset{*}{V}{}_{0i}^{e\theta}$, then, from [2, Eq. (124)],

$$\overset{*}{V}{}_{0i}^{e\theta} = \frac{j\pi\, k\Delta_p \sin v_p \cos\theta_t}{4}(F_{1a} - F_{-1a}) - \frac{\pi k\Delta_p \cos v_p \sin\theta_t}{2} F_{0a} +$$
$$(-1)^{p-i}\left(\frac{j\pi k\Delta_p \sin v_p \cos\theta_t}{4}(F_{1b} - F_{-1b}) - \frac{\pi k\Delta_p \cos v_p \sin\theta_t}{2} F_{0b}\right) \tag{98}$$

where i is either p-1 or p, but i is neither 0 nor P-1. Here, P is the number of data points on the generating curve of S. The integration from

t_p^- to t_{p+1}^- contributes to $\beta V_{0i}^{m\phi}$ of (97) only for i = p and gives all of $\beta V_{0p}^{m\phi}$. From [2, Eq. (127)], we obtain

$$\beta V_{0p}^{m\phi} = \frac{j\beta\pi k\Delta_p}{2} \left((F_{1a} - F_{-1a}) + \frac{\Delta_p \sin v_p}{2\rho_p} (F_{1b} - F_{-1b}) \right) \tag{99}$$

Consider the elements of $\beta\vec{V}_n^{m\theta}$ and $\beta\vec{V}_n^{m\phi}$ on the right-hand side of (89). In view of the integral formula

$$J_n(k\rho \sin \theta_t) = \frac{j^{-n}}{2\pi} \int_{-\pi}^{\pi} e^{j(k\rho \sin \theta_t \cos \phi + n\phi)} d\phi \tag{100}$$

for the Bessel function J_n deduced from [6, Eq. (9.1.21)], substitution of (85) into (80) gives

$$\beta V_{ni}^{m\theta} = -\frac{\beta j^n \pi k^2}{n} \int_{t_i^-}^{t_{i+2}^-} dt\, \rho\, T_i(t) \cos v\, (J_{n+1} + J_{n-1}) e^{jkz \cos \theta_t} \tag{101}$$

where

$$J_n = J_n(k\rho \sin \theta_t) \tag{102}$$

Similarly, substitution of (86) into (81) gives

$$\beta V_{ni}^{m\phi} = -\frac{\beta j^{n+1} \pi k^2}{n} \int_{t_i^-}^{t_{i+2}^-} dt\, \rho\, T_i(t) \left(\cos v \cos \theta_t (J_{n+1} - J_{n-1}) \right.$$

$$\left. - 2 j \sin v \sin \theta_t J_n \right) e^{jkz\cos \theta_t} \tag{103}$$

The contribution to $\beta V_{ni}^{m\theta}$ due to the integration from t_p^- to t_{p+1}^- is called $\beta \overset{*}{V}{}_{ni}^{m\theta}$. The contribution to $\beta V_{ni}^{m\phi}$ due to the integration from t_p^- to t_{p+1}^- is called $\beta \overset{*}{V}{}_{ni}^{m\phi}$. Now,

$$\left.\begin{aligned} v &= v_p && (104a)\\ \rho &= \rho_p + (t-t_p)\sin v_p && (104b)\\ z &= z_p + (t-t_p)\cos v_p && (104c)\\ T_i(t) &= \frac{1}{2}\left[1 + (-1)^{p-i}\frac{2(t-t_p)}{\Delta_p}\right], i=p-1,p && (104d)\end{aligned}\right\} t_p^- \le t \le t_{p+1}^-$$

where v_p, ρ_p, z_p, and t_p are, respectively, the values of v, ρ, z, and t midway between t_p^- and t_{p+1}^-. Substitution of (104) into (101) and (103) gives

$$\beta V_{ni}^{*m\theta} = -\frac{\beta j^n \pi k^2 \Delta_p \rho_p \cos v_p}{4n}\left(F_{n+1,a} + F_{n-1,a} + \frac{\Delta_p \sin v_p}{2\rho_p}(F_{n+1,b} + F_{n-1,b})\right) -$$

$$\frac{(-1)^{p-i}\beta j^n \pi k^2 \Delta_p \rho_p \cos v_p}{4n}\left(F_{n+1,b}+F_{n-1,b} + \frac{\Delta_p \sin v_p}{2\rho_p}(F_{n+1,c}+F_{n-1,c})\right) \qquad (105)$$

$$-\beta V_{ni}^{*m\phi} = \frac{\beta j^{n+1}\pi k^2 \Delta_p \rho_p \cos v_p \cos\theta_t}{4n}\left(F_{n+1,a}-F_{n-1,a} + \frac{\Delta_p \sin v_p}{2\rho_p}(F_{n+1,b}-F_{n-1,b})\right) +$$

$$\frac{\beta j^n \pi k^2 \Delta_p \rho_p \sin v_p \sin\theta_t}{2n}\left(F_{na} + \frac{\Delta_p \sin v_p}{2\rho_p}F_{nb}\right) +$$

$$(-1)^{p-i}\left(\frac{\beta j^{n+1}\pi k^2 \Delta_p \rho_p \cos v_p \cos\theta_t}{4n}\left(F_{n+1,b}-F_{n-1,b} + \frac{\Delta_p \sin v_p}{2\rho_p}(F_{n+1,c}-F_{n-1,c})\right) +\right.$$

$$\left.\frac{\beta j^n \pi k^2 \Delta_p \rho_p \sin v_p \sin\theta_t}{2n}\left(F_{nb} + \frac{\Delta_p \sin v_p}{2\rho_p}F_{nc}\right)\right) \qquad (106)$$

where

$$F_{ma} = \frac{2}{\Delta_p} \int_{\bar{t}_p}^{\bar{t}_{p+1}} J_m(k\rho \sin \theta_t)\, e^{jkz \cos \theta_t}\, dt \tag{107a}$$

$$F_{mb} = \left(\frac{2}{\Delta_p}\right)^2 \int_{\bar{t}_p}^{\bar{t}_{p+1}} (t-t_p)\, J_m(k\rho \sin \theta_t)\, e^{jkz \cos \theta_t}\, dt \tag{107b}$$

$$F_{mc} = \left(\frac{2}{\Delta_p}\right)^3 \int_{\bar{t}_p}^{\bar{t}_{p+1}} (t-t_p)^2\, J_m(k\rho \sin \theta_t)\, e^{jkz \cos \theta_t}\, dt \tag{107c}$$

In (107), $m = n-1,n,n+1$. Also, ρ and z are given by (104b) and (104c). Thanks to (87), expressions (105) and (106) can be viewed as $\beta V^{*m\theta}_{-ni}$ and $\beta V^{*m\phi}_{-ni}$, respectively.

Evaluation of (107) by means of an n_T point Gaussian quadrature formula yields

$$F_{ma} = \sum_{\ell=1}^{n_T} A_\ell^{(n_T)}\, J_m(k\, \hat{\rho}_\ell \sin \theta_t)\, e^{jk\hat{z}_\ell \cos \theta_t} \tag{108a}$$

$$F_{mb} = \sum_{\ell=1}^{n_T} A_\ell^{(n_T)}\, x_\ell^{(n_T)}\, J_m(k\, \hat{\rho}_\ell \sin \theta_t)\, e^{jk\hat{z}_\ell \cos \theta_t} \tag{108b}$$

$$F_{mc} = \sum_{\ell=1}^{n_T} A_\ell^{(n_T)}\, (x_\ell^{(n_T)})^2\, J_m(k\, \hat{\rho}_\ell \sin \theta_t)\, e^{jk\hat{z}_\ell \cos \theta_t} \tag{108c}$$

where, as in [2, Eqs. (134) and (135)],

$$\hat{\rho}_\ell = \rho_p + \frac{\Delta_p x_\ell^{(n_T)}}{2} \sin v_p \tag{109}$$

$$\hat{z}_\ell = z_p + \frac{\Delta_p x_\ell^{(n_T)}}{2} \cos v_p \tag{110}$$

The abscissas $x_\ell^{(n_T)}$ and weights $A_\ell^{(n_T)}$ in (108)-(110) are the same as in [2, Eqs. (132)-(135)].

Thanks to (74) and (75), the elements of the column vectors $\beta\vec{V}_n^{e\theta}$ and $\beta\vec{V}_n^{e\phi}$ on the right-hand side of (89) are given by

$$\beta V_{ni}^{e\theta} = \beta k\rho_i \, V_{ni}^{\phi\theta} \tag{111}$$

$$\beta V_{ni}^{e\phi} = \beta k\rho_i \, V_{ni}^{\phi\phi} \tag{112}$$

where $V_{ni}^{\phi\theta}$ and $V_{ni}^{\phi\phi}$ are given by [2, Eqs. (114) and (117)]. The integration from t_p^- to t_{p+1}^- contributes to $V_{ni}^{\phi\theta}$ only for i=p and gives all of $V_{np}^{\phi\theta}$. Upon replacement of i by p in (111) and substitution of [2, Eq. (125)] for $V_{np}^{\phi\theta}$, the negative of (111) becomes

$$-\beta V_{np}^{e\theta} = -\frac{\beta j^n \pi k^2 \Delta_p \rho_p \cos\theta_t}{2} \left(F_{n+1,a}+F_{n-1,a} + \frac{\Delta_p \sin v_p}{2\rho_p}\left(F_{n+1,b}+F_{n-1,b}\right)\right) \tag{113}$$

where F_{ma} and F_{mb} are given by (108a) and (108b). Similarly, substitution of [2, Eq. (127)] into (112) yields

$$\beta V_{np}^{e\phi} = \frac{\beta j^{n+1} \pi k^2 \Delta_p \rho_p}{2} \left(F_{n+1,a}-F_{n-1,a} + \frac{\Delta_p \sin v_p}{2\rho_p}\left(F_{n+1,b}-F_{n-1,b}\right)\right) \tag{114}$$

Thanks to (87), expressions (113) and (114) can be viewed as $\beta V_{-np}^{e\theta}$ and $\beta V_{-np}^{e\phi}$, respectively.

In the subroutine PLANE, the elements of $\hat{V}_0$ of (88) are calculated by means of (98) and (99), and the elements of $\hat{V}_n$ of (89) are calculated for negative values of n by means of (105), (106), (113), and (114). The index IP of DO loop 12 obtains p in (98), (99), (105), (106), (113), and (114). DO loop 13 puts $\frac{k}{2}\hat{\rho}_\ell$ of (109) and $k\hat{z}_\ell$ of (110) in R2(L) and Z2(L), respectively, for ℓ = L. The index K of DO loop 14 obtains the Kth angle of incidence θ_t of (93).

The index L of DO loop 15 obtains ℓ in (108). Lines 57-82 calculate S and BJ(m+2) so that

$$BJ(m+2) = S{*}J_m(k\hat{\rho}_\ell \sin \theta_t), \quad m = M1-1, M1, \ldots M2+1, \; m \neq -1 \tag{115}$$

The calculation of BJ(m+2) and S is described in [2, p. 59]. As the index L of DO loop 15 changes, line 88 accumulates F_{ma} of (108a) in FA(m+2), line 89 accumulates F_{mb} of (108b) in FB(m+2), and line 90 accumulates F_{mc} of (108c) in FC(m+2). If F_{-1a}, F_{-1b}, and F_{-1c} are needed, lines 94-96 use the formulas

$$\left.\begin{aligned} F_{-1a} &= -F_{1a} \\ F_{-1b} &= -F_{1b} \\ F_{-1c} &= -F_{1c} \end{aligned}\right\} \tag{116}$$

to store F_{-1a}, F_{-1b}, and F_{-1c} in FA(1), FB(1), and FC(1) respectively.

In DO loop 27, (98) and (99) are obtained if M is 2. If M is greater than 2, then (105), (106), (113), and (114) are obtained for n=M-2. If M is 2, then R(K1), R(K2), R(K2+MT), R(K4), R(K5), and R(K5+MT) are reserved for $\overset{*}{V}{}^{e\theta}_{0,p-1}$, $\overset{*}{V}{}^{e\theta}_{0p}$, 0, 0, 0, and $\beta V^{m\phi}_{0p}$, respectively. If M is greater than 2, then R(K1), R(K2), R(K2+MT), R(K4), R(K5), and R(K5+MT) are reserved for $\beta\overset{*}{V}{}^{m\theta}_{n,p-1}$, $\beta\overset{*}{V}{}^{m\theta}_{np}$, $-\beta V^{e\theta}_{np}$, $-\beta\overset{*}{V}{}^{m\phi}_{n,p-1}$, $-\beta\overset{*}{V}{}^{m\phi}_{np}$, and $\beta V^{e\phi}_{np}$, respectively. Table 2 describes the action of some statements in DO loop 27. The statement whose line number is given in the third column of Table 2 stores the text quantity of the second column in the variable in PLANE listed in the first column.

Table 2. Variables in PLANE versus text quantities.

Variable in PLANE	Text Quantity	Line Number
F1A	$F_{n+1,a} - F_{n-1,a}$	101
F1B	$F_{n+1,b} - F_{n-1,b}$	102
UB	$j^{n+1}\pi$	103
UC	$\frac{j\pi k\Delta_p \sin v_p \cos \theta_t}{4}$	108
U5	$-\frac{\pi k\Delta_p \cos v_p \sin \theta_t}{2}$	109
U2	$\frac{j\pi k\Delta_p \sin v_p \cos \theta_t}{4}(F_{1a}-F_{-1a}) - \frac{\pi k\Delta_p \cos v_p \sin \theta_t}{2} F_{0a}$	110
U3	$\frac{j\pi k\Delta_p \sin v_p \cos \theta_t}{4}(F_{1b}-F_{-1b}) - \frac{\pi k\Delta_p \cos v_p \sin \theta_t}{2} F_{0b}$	111
[illegible](K5+MT)	$\beta V_{0p}^{m\phi}$	115
F2A	$F_{n+1,a} + F_{n-1,a}$	117
F2B	$F_{n+1,b} + F_{n-1,b}$	118
F2C	$F_{n+1,c} + F_{n-1,c}$	119
F1C	$F_{n+1,c} - F_{n-1,c}$	120
UC	$\frac{j^n\pi}{n}$	121
U5	$-\frac{\beta j^n \pi k^2 \Delta_p \rho_p \cos v_p}{4n}$	122
U2	$-\frac{\beta j^n \pi k^2 \Delta_p \rho_p \cos v_p}{4n}(F_{n+1,a}+F_{n-1,a} + \frac{\Delta_p \sin v_p}{2\rho_p}(F_{n+1,b}+F_{n-1,b}))$	123
U3	$-\frac{\beta j^n \pi k^2 \Delta_p \rho_p \cos v_p}{4n}(F_{n+1,b}+F_{n-1,b} + \frac{\Delta_p \sin v_p}{2\rho_p}(F_{n+1,c}+F_{n-1,c}))$	124

U5	$\dfrac{\beta j^{n}\pi k^{2}\Delta_{p}\rho_{p}\sin v_{p}\sin\theta_{t}}{2n}$	125
UC	$\dfrac{\beta j^{n+1}\pi k^{2}\Delta_{p}\rho_{p}\cos v_{p}\cos\theta_{t}}{4n}$	126
U4	$\dfrac{\beta j^{n+1}\pi k^{2}\Delta_{p}\rho_{p}\cos v_{p}\cos\theta_{t}}{4n}(F_{n+1,a}-F_{n-1,a}+\dfrac{\Delta_{p}\sin v_{p}}{2\rho_{p}}(F_{n+1,b}-F_{n-1,b}))+$	
	$\dfrac{\beta j^{n}\pi k^{2}\Delta_{p}\rho_{p}\sin v_{p}\sin\theta_{t}}{2n}(F_{na}+\dfrac{\Delta_{p}\sin v_{p}}{2\rho_{p}}F_{nb})$	127
U5	$\dfrac{\beta j^{n+1}\pi k^{2}\Delta_{p}\rho_{p}\cos v_{p}\cos\theta_{t}}{4n}(F_{n+1,b}-F_{n-1,b}+\dfrac{\Delta_{p}\sin v_{p}}{2\rho_{p}}(F_{n+1,c}-F_{n-1,c}))+$	
	$\dfrac{\beta j^{n}\pi k^{2}\Delta_{p}\rho_{p}\sin v_{p}\sin\theta_{t}}{2n}(F_{nb}+\dfrac{\Delta_{p}\sin v_{p}}{2\rho_{p}}F_{nc})$	128
R(K2+MT)	$-\beta V_{np}^{e\theta}$	129
R(K5+MT)	$\beta V_{np}^{e\phi}$	130

```
C     LISTING OF THE SUBROUTINE PLANE
      SUBROUTINE PLANE(M1,M2,NF,NP,NT,RH,ZH,XT,AT,THR,R)
      COMPLEX R(240),U,U1,UA,UB,UC,FA(10),FB(10),FC(10),F2A,F2B,F2C,F1A
      COMPLEX F1B,F1C,U2,U3,U4,U5,CMPLX
      DIMENSION RH(43),ZH(43),XT(10),AT(10),THR(3),CS(3),SN(3),R2(10)
      DIMENSION Z2(10),BJ(50)
      MP=NP-1
      MT=MP-1
      N=MT+MP
      N2=2*N
      DO 11 K=1,NF
      X=THR(K)
      CS(K)=COS(X)
      SN(K)=SIN(X)
   11 CONTINUE
      U=(0.,1.)
      U1=3.141593*U**M1
      M3=M1+1
      M4=M2+3
      IF(M1.EQ.0) M3=2
      M5=M1+2
      M6=M2+2
      DO 12 IP=1,MP
      K2=IP
      I=IP+1
      DR=.5*(RH(I)-RH(IP))
      DZ=.5*(ZH(I)-ZH(IP))
      D1=SQRT(DR*DR+DZ*DZ)
      IF(IP.EQ.1) RD=1./(D1*D1)
      R3=.5*(RH(I)+RH(IP))
      DR3=DR*R3*RD
      R1=.5*R3
      D8=-DZ*R1*RD
      D1R=D1*RD
      D6=R3*D1R
      Z1=.5*(ZH(I)+ZH(IP))
      DR=.5*DR
      D2=DR/R1
      DC 13 L=1,NT
      R2(L)=R1+DR*XT(L)
      Z2(L)=Z1+DZ*XT(L)
   13 CONTINUE
      DO 14 K=1,NF
      CC=CS(K)
      SS=SN(K)
      D3=DR*CC
      D4=-DZ*SS
      D7=-D6*CC
      D9=-D8*CC
      D5=DR3*SS
      DO 23 M=M3,M4
      FA(M)=0.
      FB(M)=0.
      FC(M)=0.
   23 CONTINUE
      DO 15 L=1,NT
      X=SS*R2(L)
      IF(X.GT..5E-7) GO TO 19
      DO 20 M=M3,M4
      BJ(M)=0.
```

```
   20 CONTINUE
      BJ(2)=1.
      S=1.
      GO TO 18
   19 M=2.8*X+14.-2./X
      IF(X.LT..5) M=11.8+ALOG10(X)
      IF(M.LT.M4) M=M4
      BJ(M)=0.
      JM=M-1
      BJ(JM)=1.
      DO, 16 J=4,M
      J2=JM
      JM=JM-1
      J1=JM-1
      BJ(JM)=J1/X*BJ(J2)-BJ(JM+2)
   16 CONTINUE
      S=0.
      IF(M.LE.4) GO TO 24
      DO 17 J=4,M,2
      S=S+BJ(J)
   17 CONTINUE
   24 S=BJ(2)+2.*S
   18 ARG=Z2(L)*CC
      UA=AT(L)/S*CMPLX(COS(ARG),SIN(ARG))
      UB=XT(L)*UA
      UC=XT(L)*UB
      DO 25 M=M3,M4
      FA(M)=BJ(M)*UA+FA(M)
      FB(M)=BJ(M)*UB+FB(M)
      FC(M)=BJ(M)*UC+FC(M)
   25 CONTINUE
   15 CONTINUE
      IF(M1.NE.0) GO TO 26
      FA(1)=-FA(3)
      FB(1)=-FB(3)
      FC(1)=-FC(3)
   26 UA=U1
      DO 27 M=M5,M6
      M7=M-1
      M8=M+1
      F1A=FA(M8)-FA(M7)
      F1B=FB(M8)-FB(M7)
      UB=U*UA
      K1=K2-1
      K4=K1+N
      K5=K2+N
      IF(M.NE.2) GO TO 28
      UC=D3*UB
      U5=D4*UA
      U2=UC*F1A+U5*FA(M)
      U3=UC*F1B+U5*FB(M)
      U4=0.
      U5=0.
      R(K2+MT)=0.
      R(K5+MT)=D1R*UB*(F1A+D2*F1B)
      GO TO 29
   28 F2A=FA(M8)+FA(M7)
      F2B=FB(M8)+FB(M7)
      F2C=FC(M8)+FC(M7)
      F1C=FC(M8)-FC(M7)
      UC=(1./(M-2))*UA
      U5=D8*UC
      U2=U5*(F2A+D2*F2B)
      U3=U5*(F2B+D2*F2C)
      U5=D5*UC
      UC=D9*UC*U
      U4=UC*(F1A+D2*F1B)+U5*(FA(M)+D2*FB(M))
      U5=UC*(F1B+D2*F1C)+U5*(FB(M)+D2*FC(M))
      R(K2+MT)=D7*UA*(F2A+D2*F2B)
      R(K5+MT)=D6*UB*(F1A+D2*F1B)
   29 IF(IP.EQ.1) GO TO 21
      R(K1)=R(K1)+U2-U3
      R(K4)=R(K4)+U4-U5
      IF(IP.EQ.MP) GO TO 22
   21 R(K2)=U2+U3
      R(K5)=U4+U5
   22 K2=K2+N2
      UA=UB
   27 CONTINUE
   14 CONTINUE
   12 CONTINUE
      RETURN
      END
```

V. THE SUBROUTINES DECOMP AND SOLVE

The subroutines DECOMP(N, IPS, UL) and SOLVE(N, IPS, UL, B, X) solve a system of N linear equations in N unknowns. The input to DECOMP consists of N and the N by N matrix of coefficients on the left-hand side of the matrix equation stored by columns in UL. The output from DECOMP is IPS and UL. This output is fed into SOLVE. The rest of the input to SOLVE consists of N and the column of coefficients on the right-hand side of the matrix equation stored in B. SOLVE puts the solution to the matrix equation in X.

Minimum allocations are given by

```
COMPLEX UL(N*N)
DIMENSION SCL(N), IPS(N)
```

in DECOMP and by

```
COMPLEX UL(N*N), B(N), X(N)
DIMENSION IPS(N)
```

in SOLVE.

More detail concerning DECOMP and SOLVE is on pages 46-49 of [7].

```
C     LISTING OF THE SUBROUTINES DECOMP AND SOLVE
      SUBROUTINE DECOMP(N,IPS,UL)
      COMPLEX UL(1600),PIVOT,EM
      DIMENSION SCL(40),IPS(40)
      DO 5 I=1,N
      IPS(I)=I
      RN=0.
      J1=I
      DO 2 J=1,N
      ULM=ABS(REAL(UL(J1)))+ABS(AIMAG(UL(J1)))
      J1=J1+N
      IF(RN-ULM) 1,2,2
    1 RN=ULM
    2 CONTINUE
      SCL(I)=1./RN
    5 CONTINUE
      NM1=N-1
      K2=0
      DO 17 K=1,NM1
      BIG=0.
      DO 11 I=K,N
      IP=IPS(I)
      IPK=IP+K2
      SIZE=(ABS(REAL(UL(IPK)))+ABS(AIMAG(UL(IPK))))*SCL(IP)
      IF(SIZE-BIG) 11,11,10
   10 BIG=SIZE
      IPV=I
   11 CONTINUE
      IF(IPV-K) 14,15,14
   14 J=IPS(K)
      IPS(K)=IPS(IPV)
      IPS(IPV)=J
   15 KPP=IPS(K)+K2
      PIVOT=UL(KPP)
      KP1=K+1
      DO 16 I=KP1,N
      KP=KPP
      IP=IPS(I)+K2
      EM=-UL(IP)/PIVOT
   18 UL(IP)=-EM
      DO 16 J=KP1,N
      IP=IP+N
      KP=KP+N
      UL(IP)=UL(IP)+EM*UL(KP)
   16 CONTINUE
      K2=K2+N
   17 CONTINUE
      RETURN
      END
      SUBROUTINE SOLVE(N,IPS,UL,B,X)
      COMPLEX UL(1600),B(40),X(40),SUM
      DIMENSION IPS(40)
      NP1=N+1
      IP=IPS(1)
      X(1)=B(IP)
      DO 2 I=2,N
      IP=IPS(I)
      IPB=IP
      IM1=I-1
      SUM=0.
      DO 1 J=1,IM1
      SUM=SUM+UL(IP)*X(J)
    1 IP=IP+N
    2 X(I)=B(IPB)-SUM
      K2=N*(N-1)
      IP=IPS(N)+K2
      X(N)=X(N)/UL(IP)
      DO 4 IBACK=2,N
      I=NP1-IBACK
      K2=K2-N
      IPI=IPS(I)+K2
      IP1=I+1
      SUM=0.
      IP=IPI
      DO 3 J=IP1,N
      IP=IP+N
    3 SUM=SUM+UL(IP)*X(J)
    4 X(I)=(X(I)-SUM)/UL(IPI)
      RETURN
      END
```

VI. THE MAIN PROGRAM FOR THE NEW E-FIELD SOLUTION

The main program for the new E-field solution uses the subroutines ZMAT, PLANE, DECOMP, and SOLVE to calculate the surface density of electric current $\underline{J}$ and electric charge q_e induced on S by an axially incident plane wave. For the surface S of revolution illuminated by the θ-polarized plane wave (60), [1, Eq. (46)] specializes to

$$\underline{J} = \sum_{n=-\infty}^{\infty} \left(\sum_{j=1}^{N_m} I_{nj}^{m\theta} \underline{J}_{nj}^{m} + \sum_{j=1}^{N_e} I_{nj}^{e\theta} k\rho_j \underline{J}_{nj}^{e} \right) \tag{117}$$

where the expansion functions $\underline{J}_{nj}^{m}$ and $k\rho_j \underline{J}_{nj}^{e}$ are given by [1, Eqs. (91), (92), (100), and (101)]. According to (90), the coefficients $I_{nj}^{m\theta}$ and $I_{nj}^{e\theta}$ in (117) are the elements of the column vector $\vec{I}_n^{\theta}$ that satisfies

$$\hat{Z}_n \vec{I}_n^{\theta} = \vec{V}_n^{\theta} \quad , \quad n = 0, \pm 1, \pm 2, \ldots \tag{118}$$

where

$$\vec{I}_0^{\theta} = \begin{bmatrix} \vec{I}_0^{e\theta} \\ \\ 0 \end{bmatrix} \tag{119}$$

$$\vec{I}_n^{\theta} = \begin{bmatrix} \vec{I}_n^{m\theta} \\ \\ \vec{I}_n^{e\theta} \end{bmatrix} \quad , \quad n = \pm 1, \pm 2, \ldots \tag{120}$$

and

$$\vec{V}_0^{\theta} = \begin{bmatrix} \vec{V}_0^{e\theta} \\ \\ 0 \end{bmatrix} \tag{121}$$

$$\vec{V}_n^{\theta} = \beta \begin{bmatrix} \vec{V}_n^{m\theta} \\ \\ \vec{V}_n^{e\theta} \end{bmatrix} \quad n = \pm 1, \pm 2, \ldots \tag{122}$$

It is evident from (15), (2), and (13) that

$$\hat{Z}_{-n} = \beta \begin{bmatrix} Z_n^{mm} & -Z_n^{me} \\ -Z_n^{em} & Z_n^{ee} \end{bmatrix} , \quad n = 1,2,... \tag{123}$$

Equations (87) and (122) imply that

$$\vec{V}^{\theta}_{-n} = \beta \begin{bmatrix} \vec{V}_n^{m\theta} \\ -\vec{V}_n^{e\theta} \end{bmatrix} , \quad n = 1,2,... \tag{124}$$

In view of (123) and (124), comparison of (118) with (118) with n replaced by -n reveals that

$$\vec{I}^{\theta}_{-n} = \begin{bmatrix} \vec{I}_n^{m\theta} \\ -\vec{I}^{e\theta} \end{bmatrix} , \quad n = 1,2,... \tag{125}$$

It is assumed that the angle θ_t of incidence of the plane wave is either zero or π radians. Consequently, (98), (105), and (113) predict that

$$\vec{V}^{\theta}_n = 0, \quad n \neq \pm 1 \tag{126}$$

Substitution of (126) into (118) gives

$$\vec{I}^{\theta}_n = 0, \quad n \neq \pm 1 \tag{127}$$

Equations (125) and (127) reduce (117) to

$$\underline{J} = \sum_{j=1}^{N_m} I_{1j}^{m\theta}(\underline{J}_{1j}^m + \underline{J}_{-1j}^m) + \sum_{j=1}^{N_e} I_{1j}^{e\theta}\, k\rho_j(\underline{J}_{1j}^e - \underline{J}_{-1j}^e) \tag{128}$$

where $I_{1j}^{m\theta}$ is the jth element of $\vec{I}_1^{m\theta}$ and $I_{1j}^{e\theta}$ is the jth element of $\vec{I}_1^{e\theta}$. According to (120), the combination of $\vec{I}_1^{m\theta}$ and $\vec{I}_1^{e\theta}$ is the solution $\vec{I}_1^{\theta}$ of (118) for n = 1.

The expansion functions in (128) are given by [1, Eqs. (100) and (101)] where $\underline{J}^{t}_{nj}$ and $\underline{J}^{\phi}_{nj}$ are given by [1, Eqs. (82) and (83)]. As a result, (128) reduces to

$$\underline{J} = \underline{u}_t J_t k \cos\phi + \underline{u}_\phi (J^m_\phi + J^e_\phi)\, k \sin\phi \tag{129}$$

where

$$J_t = 2 \sum_{j=1}^{P-2} \left(\frac{T_j(t)}{k\rho}\right) I^{m\theta}_{1j} \tag{130}$$

$$J^m_\phi = 2 \sum_{j=1}^{P-2} \left(\frac{P_{j+1}(t)}{k\Delta_{j+1}} - \frac{P_j(t)}{k\Delta_j}\right) I^{m\theta}_{1j} \tag{131}$$

$$J^e_\phi = 2j \sum_{j=1}^{P-1} P_j(t)\, I^{e\theta}_{1j} \tag{132}$$

In (129), J^m_ϕ is due to the magnetostatic expansion functions $\underline{J}^m_{1j}$ and $\underline{J}^m_{-1j}$ in (128), and J^e_ϕ is due to the electrostatic expansion functions $\underline{J}^e_{1j}$ and $\underline{J}^e_{-1j}$ in (128). Expression (131) can be rewritten as

$$J^m_\phi = 2 \sum_{j=1}^{P-1} \frac{P_j(t)}{k\Delta_j} \left(I^{m\theta}_{1,j-1} - I^{m\theta}_{1j}\right) \tag{133}$$

where

$$\left.\begin{aligned} I^{m\theta}_{10} &= 0 \\ I^{m\theta}_{1,P-1} &= 0 \end{aligned}\right\} \tag{134}$$

The electric charge q_e is given by [1, Eq. (1)] where $\nabla_s \cdot$ and $\underline{J}$ are given by [1, Eq. (B-3)] and (129), respectively. Thus,

$$q_e = \frac{1}{-j\omega} \left(\frac{1}{\rho} \frac{\partial}{\partial t} (\rho\, J_t k \cos\phi) + \frac{1}{\rho} \frac{\partial}{\partial\phi} ((J^m_\phi + J^e_\phi)\, k \sin\phi)\right) \tag{135}$$

Thanks to expressions (130)-(132) for J_t, J^m_ϕ, and J^e_ϕ, (135) becomes

$$q_e = q(\frac{k}{c}) \cos \phi \tag{136}$$

where

$$q = -2 \sum_{j=1}^{P-1} (\frac{P_j(t)}{k\rho}) I_{1j}^{e\theta} \tag{137}$$

In (136), c is the speed of light.

As a result of the development (117)-(137), $\underline{J}$ is given by (129) where J_t, J_ϕ^m, and J_ϕ^e are given by (130), (133), and (132), respectively. q_e is given by (136) where q is given by (137). The coefficients $I_{1j}^{m\theta}$ and $I_{1j}^{e\theta}$ in (130), (133), (132), and (137) are the jth elements of $\vec{I}_1^{m\theta}$ and $\vec{I}_1^{e\theta}$, respectively. Equation (120) gives

$$\begin{bmatrix} \vec{I}_1^{m\theta} \\ \\ \vec{I}_1^{e\theta} \end{bmatrix} = \vec{I}_1^{\theta} \tag{138}$$

According to (118), $\vec{I}_1^{\theta}$ satisfies

$$\hat{Z}_1 \vec{I}_1^{\theta} = \vec{V}_1^{\theta} \tag{139}$$

As given by (129), $\underline{J}$ is the electric current induced on S by the θ-polarized plane wave electric field $\underline{E}^\theta$ of (60) with $\theta_t = 0$ or $\theta_t = \pi$ radians. If θ_t is zero, (60) reduces to

$$\underline{E}^\theta = \underline{u}_x k\eta e^{jkz} \tag{140}$$

If θ_t is π radians, (60) reduces to

$$\underline{E}^\theta = -\underline{u}_x k\eta e^{-jkz} \tag{141}$$

Hence, $\underline{J}$ of (129) is the electric current induced on S by the incident electric field given by either (140) or (141).

The main program for the new E-field solution is listed at the end of this section. In this main program, input data are read from punched cards according to

```
      READ(1,15) NT, NPHI
15    FORMAT(2I3)
      READ(1,10)(XT(K), K=1, NT)
      READ(1,10)(AT(K), K=1, NT)
10    FORMAT(5E14.7)
      READ(1,10)(X(K), K=1, NPHI)
      READ(1,10)(A(K), K=1, NPHI)
      READ(1, 16) NP, BK, THR(1)
16    FORMAT(I3, 2E14.7)
      READ(1,18)(RH(I), I=1, NP)
      READ(1, 18)(ZH(I), I=1, NP)
18    FORMAT(10F8.4)
```

NT, NPHI, XT, AT, X, and A are Gaussian quadrature data that are fed into the subroutine ZMAT. NT is n_t in [2, Eqs. (62)-(63)]. The variable n_T that appears in [2, Eqs. (132)-(133)] was chosen to be equal to n_t. Hence, NT can also be viewed as n_T. XT contains the n_t abscissas $x_\ell^{(n_t)}$, in [2, Eqs. (62)-(63)], and AT contains the n_t weights $A_\ell^{(n_t)}$, in [2, Eqs. (62)-(63)]. NPHI is n_ϕ in [2, Eqs. (64)-(66)]. X contains the n_ϕ abscissas $x_\ell^{(n_\phi)}$ of [2, Eq. (70)], and A contains the n_ϕ weights $A_\ell^{(n_\phi)}$ in [2, Eqs. (64)-(66)]. NP is the number P of data points on the generating curve of S. P appears in [1, Eqs. (100) and (101)]. BK is the propagation constant k in 1/meters. k appears in (140) and (141). THR(1) is the angle of incidence θ_t of (62)-(63) that is fed into the subroutine PLANE. THR(1) is in

radians. THR(1) should be either zero or π. If THR(1) is zero, the incident electric field is given by (140). If THR(1) is π, the incident electric field is given by (141). RH and ZH contain the coordinates of the data points on the generating curve of S.

$$RH(J) = \rho(t_J^-), \ J = 1,2,\ldots NP \tag{142}$$

$$ZH(J) = z(t_J^-), \ J = 1,2,\ldots NP \tag{143}$$

Here, z is the coordinate along the axis about which the generating curve of S is rotated, and ρ is the distance from this axis. Also, (t_J^-) denotes evaluation at the Jth data point. In (142)-(143), $\rho(t_J^-)$ and $z(t_J^-)$ are in meters. The main program for the new E-field solution uses the subprograms ZMAT, BLOG, PLANE, DECOMP, and SOLVE of Sections II, III, IV, and V.

Minimum allocations in this main program are given by

```
COMPLEX Z(N*N), R(2*N), B(N), C(N), CE(NP-1),
    CQ(NP-1)
DIMENSION RH(NP), ZH(NP), X(NPHI), A(NPHI),
    XT(NT), AT(NT), IPS(N)
```

where

$$N = 2*NP-3 \tag{144}$$

With regard to (129) and (136), the main program prints J_t under the heading "T COMPONENT OF ELECTRIC CURRENT," $J_\phi^m + J_\phi^e$ under the heading "PHI COMPONENT OF ELECTRIC CURRENT," J_ϕ^m under the heading "MAGNETOSTATIC PART OF JP," J_ϕ^e under the heading "ELECTROSTATIC PART OF JP," and q under the heading "ELECTRIC CHARGE." A real part, an imaginary part, and a magnitude are printed on each line. The jth line of printed output for J_t is the value of J_t at the peak of $T_j(t)$. This peak occurs at

$$t = t^-_{j+1} \tag{145}$$

The jth line of printed output for each of $J^m_\phi + J^e_\phi$, J^m_ϕ, J^e_ϕ, and q is the value at the center of the domain of $P_j(t)$. This center occurs at

$$t = \frac{1}{2}\,(t^-_j + t^-_{j+1}) \tag{146}$$

The sample input and output data are for the conducting circular disk of radius 0.002 wavelengths illuminated by the incident electric field (140). The disk is placed in the xy plane and is centered at the origin. The factor k in (129) and (136) compensates for the factor k in (140) so that J_t, $J^m_\phi + J^e_\phi$, and q coincide with the variables J_t, J_ϕ, and q used in [1, Eqs. (108) and (109)]. The printed output for $|J_t|$ is plotted by means of the interior ×'s in [1, Fig. 1]. The printed output for $|J^m_\phi + J^e_\phi|$ is plotted with the ×'s in [1, Fig. 2]. The printed output for $|q|$ is plotted with the ×'s in [1, Fig. 3].

DO loop 28 multiplies RH(J) and ZH(J) by the propagation constant k. With regard to (139), line 41 puts $\hat{Z}_1$ in Z. Line 44 prints out the first column of $\hat{Z}_1$. Line 46 puts $\vec{V}^{\theta}_{-1}$ of (122) in R(1) through R(N) where N is given by (144). Line 46 also puts the column vectors $\beta\vec{V}^{m\phi}_{-1}$ and $\beta\vec{V}^{e\phi}_{-1}$ of (89) in R(N+1) through R(2*N), but these column vectors are not used in the main program. DO loop 22 and line 52 take advantage of (124) to store $\vec{V}^{\theta}_1$ in B(1) through B(N). $\vec{V}^{\theta}_1$ is needed because it appears in (139). Lines 55 and 56 put the solution $\vec{I}^{\theta}_1$ of (139) in C(1) through C(N).

DO loop 24 prints out J_t of (130). Inside DO loop 27, line 85 puts J^e_ϕ of (132) in CE(J), line 86 puts q of (137) in CQ(J), and

line 90 prints out $J_\phi^m + J_\phi^e$ and J_ϕ^m of (133). As intermediate steps, line 79 puts $2/(k\Delta_J)$ in C4, lines 80-83 put J_ϕ^m of (133) in C1, and line 87 puts $J_\phi^m + J_\phi^e$ in C3. Inside DO loop 32, line 103 prints out J_ϕ^e of (132) and q of (137).

```
-ISTING OF THE MAIN PROGRAM FOR THE NEW E-FIELD SOLUTION
THE SUBPROGRAMS ZMAT, BLOG, PLANE, DECOMP, AND SOLVE ARE NEEDED
JOB (XXXX,XXXX,1,1),'MAUTZ,JCE',REGION=200K
C WATFIV
YSIN DD *
        MAUTZ,TIME=5,PAGES=60
COMPLEX Z(1600),R(240),B(40),C(40),U,C1,C3,CE(20),CQ(20)
DIMENSION THR(3),RH(43),ZH(43),X(48),A(48),XT(10),AT(10),IPS(40)
READ(1,15) NT,NPHI
FORMAT(2I3)
WRITE(3,30) NT,NPHI
FORMAT('  NT NPHI'/1X,I3,I5)
READ(1,10)(XT(K),K=1,NT)
READ(1,10)(AT(K),K=1,NT)
FORMAT(5E14.7)
WRITE(3,11)(XT(K),K=1,NT)
WRITE(3,12)(AT(K),K=1,NT)
FORMAT(' XT'/(1X,5E14.7))
FORMAT(' AT'/(1X,5E14.7))
READ(1,10)(X(K),K=1,NPHI)
READ(1,10)(A(K),K=1,NPHI)
WRITE(3,13)(X(K),K=1,NPHI)
WRITE(3,14)(A(K),K=1,NPHI)
FORMAT(' X'/(1X,5E14.7))
FORMAT(' A'/(1X,5E14.7))
READ(1,16) NP,BK,THR(1)
FORMAT(I3,2E14.7)
WRITE(3,17) NP,BK,THR(1)
FORMAT('  NP',6X,'BK',12X,'THR'/1X,I3,2E14.7)
READ(1,18)(RH(I),I=1,NP)
READ(1,18)(ZH(I),I=1,NP)
FORMAT(10F8.4)
WRITE(3,19)(RH(I),I=1,NP)
WRITE(3,20)(ZH(I),I=1,NP)
FORMAT(' RH'/(1X,10F8.4))
FORMAT(' ZH'/(1X,10F8.4))
DO 28 J=1,NP
RH(J)=BK*RH(J)
ZH(J)=BK*ZH(J)
CONTINUE
CALL ZMAT(1,1,NP,NPHI,NT,RH,ZH,X,A,XT,AT,Z)
MT=NP-2
N=2*MT+1
WRITE(3,29)(Z(J),J=1,N)
FORMAT(' Z'/(1X,6E11.4))
CALL PLANE(1,1,1,NP,NT,RH,ZH,XT,AT,THR,R)
DO 22 J=1,MT
B(J)=R(J)
J1=J+MT
B(J1)=-R(J1)
ONTINUE
(N)=-R(N)
RITE(3,23)(B(J),J=1,N)
ORMAT(' B'/(1X,6E11.4))
ALL DECOMP(N,IPS,Z)
ALL SOLVE(N,IPS,Z,B,C)
RITE(3,37)
ORMAT('0 T COMPONENT OF ELECTRIC CURRENT')
RITE(3,21)
ORMAT('    REAL JT     IMAG JT      MAG JT')
```

```
      DO 24 J=1,MT
      C1=2./RH(J+1)*C(J)
      C2=CABS(C1)
      WRITE(3,25) C1,C2
   25 FORMAT(1X,3E11.4)
   24 CONTINUE
      U=(0.,2.)
      WRITE(3,26)
   26 FORMAT('0PHI COMPONENT OF ELECTRIC CURRENT',7X,'MAGNETOSTATIC PART
     1 OF JP')
      WRITE(3,35)
   35 FORMAT('    REAL JP     IMAG JP      MAG JP',6X,'REAL JP     IMAG JP
     1      MAG JP')
      MP=NP-1
      DO 27 J=1,MP
      JP=J+1
      D1=RH(JP)-RH(J)
      D2=ZH(JP)-ZH(J)
      C4=2./SQRT(D1*D1+D2*D2)
      C1=0.
      IF(J.NE.1) C1=C1+C(J-1)
      IF(J.NE.MP) C1=C1-C(J)
      C1=C4*C1
      C3=C(J+MT)
      CE(J)=U*C3
      CQ(J)=-4./(RH(JP)+RH(J))*C3
      C3=C1+CE(J)
      C4=CABS(C3)
      C2=CABS(C1)
      WRITE(3,33) C3,C4,C1,C2
   33 FORMAT(1X,3E11.4,2X,3E11.4)
   27 CONTINUE
      WRITE(3,31)
   31 FORMAT('0',5X,'ELECTROSTATIC PART OF JP',15X,'ELECTRIC CHARGE')
      WRITE(3,36)
   36 FORMAT('    REAL JP     IMAG JP      MAG JP',6X,'REAL Q      IMAG Q
     1      MAG Q')
      DO 32 J=1,MP
      C1=CE(J)
      C2=CABS(C1)
      C3=CQ(J)
      C4=CABS(C3)
      WRITE(3,33) C1,C2,C3,C4
   32 CONTINUE
      STOP
      END
$DATA
  2 20
-0.5773503E+00 0.5773503E+00
 0.1000000E+01 0.1000000E+01
-0.9931286E+00-0.9639719E+00-0.9122344E+00-0.8391170E+00-0.7463319E+00
-0.6360537E+00-0.5108670E+00-0.3737061E+00-0.2277859E+00-0.7652652E-01
 0.7652652E-01 0.2277859E+00 0.3737061E+00 0.5108670E+00 0.6360537E+00
 0.7463319E+00 0.8391170E+00 0.9122344E+00 0.9639719E+00 0.9931286E+00
 0.1761401E-01 0.4060143E-01 0.6267205E-01 0.8327674E-01 0.1019301E+00
 0.1181945E+00 0.1316886E+00 0.1420961E+00 0.1491730E+00 0.1527534E+00
 0.1527534E+00 0.1491730E+00 0.1420961E+00 0.1316886E+00 0.1181945E+00
 0.1019301E+00 0.8327674E-01 0.6267205E-01 0.4060143E-01 0.1761401E-01
 16 0.8377580E-03 0.0000000E+00
 0.0000  1.0000  2.0000  3.0000  4.0000  5.0000  6.0000  7.0000  8.0000  9.0000
```

```
10.0000 11.0000 12.0000 13.0000 14.0000 15.0000
 0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
 0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
$STOP
/*
//

PRINTED OUTPUT
 NT NPHI
  2   20
XT
 0.5773503E+00 0.5773503E+00
AT
 0.1000000E+01 0.1000000E+01
X
-0.9931286E+00-0.9639719E+00-0.9122344E+00-0.8391170E+00-0.7463319E+00
-0.6360537E+00-0.5108670E+00-0.3737061E+00-0.2277859E+00-0.7652652E-01
 0.7652652E-01 0.2277859E+00 0.3737061E+00 0.5108670E+00 0.6360537E+00
 0.7463319E+00 0.8391170E+00 0.9122344E+00 0.9639719E+00 0.9931286E+00
A
 0.1761401E-01 0.4060143E-01 0.6267208E-01 0.8327675E-01 0.1019301E+00
 0.1181945E+00 0.1316886E+00 0.1420961E+00 0.1491730E+00 0.1527534E+00
 0.1527534E+00 0.1491730E+00 0.1420961E+00 0.1316886E+00 0.1181945E+00
 0.1019301E+00 0.8327675E-01 0.6267208E-01 0.4060143E-01 0.1761401E-01
 NP      BK              THR
 16 0.8377580E-03 0.0000000E+00
PH
  0.0000  1.0000  2.0000  3.0000  4.0000  5.0000  6.0000  7.0000  8.0000  9.0000
 10.0000 11.0000 12.0000 13.0000 14.0000 15.0000
TH
  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000

 0.4101E-04 0.2046E+05-0.3719E-04-0.3226E+04 0.1526E-04-0.1689E+04
 0.3052E-04-0.5296E+03-0.4578E-04-0.2448E+03-0.4578E-04-0.1347E+03
 0.0000E+00-0.8246E+02 0.6104E-04-0.5424E+02-0.1526E-03-0.3764E+02
 0.3052E-04-0.2721E+02-0.1526E-04-0.2030E+02-0.1526E-03-0.1558E+02
 0.1221E-03-0.1220E+02-0.9155E-04-0.9734E+01-0.3650E+01 0.8615E-08
-0.7732E+01 0.2608E-07 0.3126E+01 0.4098E-07 0.1353E+01 0.1490E-07
 0.7767E+00 0.4098E-07 0.5074E+00 0.7078E-07 0.3584E+00 0.8196E-07
 0.2670E+00 0.6333E-07 0.2067E+00 0.0000E+00 0.1649E+00 0.1229E-06
 0.1346E+00 0.8941E-07 0.1120E+00 0.1006E-06 0.9462E-01 0.2384E-06
 0.8102E-01 0.1192E-06 0.7016E-01 0.1788E-06

 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00
 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00
 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00
 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00
 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00-0.0000E+00 0.6283E+01
 0.0000E+00 0.1885E+02-0.0000E+00 0.3142E+02-0.0000E+00 0.4398E+02
 0.0000E+00 0.5655E+02-0.0000E+00 0.6912E+02-0.0000E+00 0.8168E+02
 0.0000E+00 0.9425E+02-0.0000E+00 0.1068E+03-0.0000E+00 0.1194E+03
 0.0000E+00 0.1319E+03-0.0000E+00 0.1445E+03-0.0000E+00 0.1571E+03
 0.0000E+00 0.1696E+03-0.0000E+00 0.1822E+03

T COMPONENT OF ELECTRIC CURRENT
  REAL JT     IMAG JT     MAG JT
0.7513E-08 0.2116E-01 0.2116E-01
0.7133E-08 0.2095E-01 0.2095E-01
0.8226E-08 0.2069E-01 0.2069E-01
```

0.8463E-08	0.2034E-01	0.2034E-01
0.8348E-08	0.1988E-01	0.1988E-01
0.7887E-08	0.1931E-01	0.1931E-01
0.7588E-08	0.1862E-01	0.1862E-01
0.7201E-08	0.1778E-01	0.1778E-01
0.6682E-08	0.1679E-01	0.1679E-01
0.6318E-08	0.1560E-01	0.1560E-01
0.5688E-08	0.1418E-01	0.1418E-01
0.4960E-08	0.1244E-01	0.1244E-01
0.4143E-08	0.1023E-01	0.1023E-01
0.2985E-08	0.7317E-02	0.7317E-02

PHI COMPONENT OF ELECTRIC CURRENT

REAL JP	IMAG JP	MAG JP
-0.7453E-08	-0.2119E-01	0.2119E-01
-0.6943E-08	-0.2106E-01	0.2106E-01
-0.1079E-07	-0.2107E-01	0.2107E-01
-0.1020E-07	-0.2107E-01	0.2107E-01
-0.9356E-08	-0.2110E-01	0.2110E-01
-0.7953E-08	-0.2112E-01	0.2112E-01
-0.9114E-08	-0.2120E-01	0.2120E-01
-0.1004E-07	-0.2130E-01	0.2130E-01
-0.8794E-08	-0.2148E-01	0.2148E-01
-0.1231E-07	-0.2179E-01	0.2179E-01
-0.1114E-07	-0.2232E-01	0.2232E-01
-0.1310E-07	-0.2326E-01	0.2326E-01
-0.1698E-07	-0.2512E-01	0.2512E-01
-0.1809E-07	-0.2828E-01	0.2828E-01
-0.4586E-07	-0.5684E-01	0.5684E-01

MAGNETOSTATIC PART OF JP

REAL JP	IMAG JP	MAG JP
-0.7513E-08	-0.2116E-01	0.2116E-01
-0.6752E-08	-0.2075E-01	0.2075E-01
-0.1041E-07	-0.2017E-01	0.2017E-01
-0.9172E-08	-0.1927E-01	0.1927E-01
-0.7889E-08	-0.1806E-01	0.1806E-01
-0.5582E-08	-0.1645E-01	0.1645E-01
-0.5795E-08	-0.1446E-01	0.1446E-01
-0.4488E-08	-0.1194E-01	0.1194E-01
-0.2536E-08	-0.8832E-02	0.8832E-02
-0.3040E-08	-0.4939E-02	0.4939E-02
0.6128E-09	0.4217E-04	0.4217E-04
0.3051E-08	0.6673E-02	0.6673E-02
0.5654E-08	0.1632E-01	0.1632E-01
0.1207E-07	0.3056E-01	0.3056E-01
0.4179E-07	0.1024E+00	0.1024E+00

ELECTROSTATIC PART OF JP

REAL JP	IMAG JP	MAG JP
0.5986E-10	-0.3437E-04	0.3437E-04
-0.1910E-09	-0.3146E-03	0.3146E-03
-0.3801E-09	-0.9001E-03	0.9001E-03
-0.1025E-08	-0.1798E-02	0.1798E-02
-0.1467E-08	-0.3044E-02	0.3044E-02
-0.2371E-08	-0.4667E-02	0.4667E-02
-0.3318E-08	-0.6742E-02	0.6742E-02
-0.5552E-08	-0.9353E-02	0.9353E-02
-0.6258E-08	-0.1265E-01	0.1265E-01
-0.9266E-08	-0.1685E-01	0.1685E-01
-0.1176E-07	-0.2236E-01	0.2236E-01
-0.1615E-07	-0.2993E-01	0.2993E-01
-0.2263E-07	-0.4143E-01	0.4143E-01
-0.3016E-07	-0.5884E-01	0.5884E-01
-0.8765E-07	-0.1593E+00	0.1593E+00

ELECTRIC CHARGE

REAL Q	IMAG Q	MAG Q
0.8204E-01	0.1429E-06	0.8204E-01
0.2503E+00	-0.1520E-06	0.2503E+00
0.4298E+00	-0.1815E-06	0.4298E+00
0.6133E+00	-0.3496E-06	0.6133E+00
0.8074E+00	-0.3891E-06	0.8074E+00
0.1013E+01	-0.5147E-06	0.1013E+01
0.1238E+01	-0.6093E-06	0.1238E+01
0.1489E+01	-0.8836E-06	0.1489E+01
0.1776E+01	-0.8789E-06	0.1776E+01
0.2117E+01	-0.1164E-05	0.2117E+01
0.2542E+01	-0.1336E-05	0.2542E+01
0.3107E+01	-0.1677E-05	0.3107E+01
0.3957E+01	-0.2161E-05	0.3957E+01
0.5203E+01	-0.2667E-05	0.5203E+01
0.1311E+02	-0.7215E-05	0.1311E+02

VII. BOUWKAMP'S POWER SERIES SOLUTION FOR A CIRCULAR DISK

The known solutions for the electric current and electric charge on a conducting circular disk excited by an axially incident plane wave are plotted in [1, Figs. 1,2, and 3]. The known solution for the electric current was obtained from Bouwkamp's formulas [3, Eqs. (23), (38), (39), Table I, and Table II]. Actually, Bouwkamp's formulas are for a disk of unit radius. They had to be changed to mksc units [4, p. 1] for a disk of radius a meters before being used. The electric charge was obtained from the electric current via the equation of continuity. The foregoing electric current and electric charge were calculated by means of a computer program. This program is listed at the end of this section.

In the present report, $e^{j\omega t}$ time dependence is assumed. However, Bouwkamp uses $e^{-i\omega t}$ time dependence. His formulas can be made valid for $e^{j\omega t}$ time dependence by replacing i by -j everywhere. Henceforth in the present report, Bouwkamp's formulas will be taken not as they stand but with i replaced by -j everywhere.

If the electric field $\underline{E}^{inc}$ given by

$$\underline{E}^{inc} = \underline{u}_x \eta\, e^{-jkz} \tag{147}$$

is incident upon a conducting circular disk of radius a meters lying in the xy plane and centered at the origin, then the electric field integral equation for the surface density $\underline{J}$ of electric current induced on the disk is

$$\underline{u}_x = j[k\, \underline{A}(\underline{r}) + \frac{1}{k} \nabla(\nabla \cdot \underline{A}(\underline{r}))]_{tan}, \quad \underline{r} \text{ on } S \tag{148}$$

where

$$\underline{A}(\underline{r}) = \frac{1}{4\pi} \iint_S \frac{\underline{J}(k,\underline{r}')\, e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} ds' \tag{149}$$

The subscript tan denotes the component in the xy plane, S is the surface of the disk, and ds' is the differential element of surface area at $\underline{r}'$. All quantities in (147)-(149) are in mksc units. We want to solve (148) and (149) for $\underline{J}$.

On the other hand, it is evident from [3, Sec. 2] that Bouwkamp considers

$$\underline{u}_x = j[k_1 \underline{A}_1(\underline{r}_1) + \frac{1}{k_1} \nabla_1(\nabla_1 \cdot \underline{A}_1(\underline{r}_1))]_{tan}\ , \ \underline{r}_1 \text{ on } S_1 \tag{150}$$

where

$$\underline{A}_1(\underline{r}_1) = \frac{1}{c} \iint_{S_1} \frac{\underline{I}(k_1, \underline{r}_1')\, e^{-jk_1|\underline{r}_1-\underline{r}_1'|}}{|\underline{r}_1-\underline{r}_1'|} ds_1' \tag{151}$$

He solves (150) and (151) for $\underline{I}$. Here, $\underline{r}_1$ and $\underline{r}_1'$ are dimensionless radius vectors, k_1 is the dimensionless wave number, ∇_1 is the ∇ operator with respect to the coordinates of $\underline{r}_1$, ds_1' is the differential element of surface area at $\underline{r}_1'$, S_1 is the surface of a disk of radius unity, and c is the speed of light.

Substitution of

$$\underline{r}_1 = \frac{\underline{r}}{a} \tag{152}$$

$$k_1 = ka \tag{153}$$

into (150) and (151) gives

$$\underline{u}_x = j[ka\ \underline{A}_1(\frac{\underline{r}}{a}) + \frac{a}{k} \nabla(\nabla \cdot \underline{A}_1(\frac{\underline{r}}{a}))]_{tan},\ \ \underline{r} \text{ on } S \tag{154}$$

where

$$\underline{A}_1(\frac{\underline{r}}{a}) = \frac{a}{c} \iint_{S_1} \frac{\underline{I}(ka, \underline{r}_1')\, e^{-jk|\underline{r}-a\,\underline{r}_1'|}}{|\underline{r}-a\underline{r}_1'|} ds_1' \tag{155}$$

In (154), ∇ operates on the coordinates of $\underline{r}$. Introduction of a new radius vector $\underline{r}'$ defined by

$$\underline{r}' = a\underline{r}_1' \tag{156}$$

reduces (155) to

$$\underline{A}_1\left(\frac{\underline{r}}{a}\right) = \frac{1}{ac} \iint_S \frac{\underline{I}\left(ka, \frac{\underline{r}'}{a}\right) e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \, ds' \tag{157}$$

Equations (154) and (157) can be rewritten as

$$\underline{u}_x = j\left[k\,\underline{A}(\underline{r}) + \frac{1}{k}\nabla(\nabla \cdot \underline{A}(\underline{r}))\right]_{tan}, \quad \underline{r} \text{ on } S \tag{158}$$

where

$$\underline{A}(\underline{r}) = \frac{1}{4\pi} \iint_S \frac{\frac{4\pi}{c}\,\underline{I}\left(ka, \frac{\underline{r}'}{a}\right) e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \, ds' \tag{159}$$

Equations (158) and (159) were derived from (150) and (151) for a disk of unit radius. However, (158) and (159) will coincide with (148) and (149) for the disk of radius a if

$$\underline{J}(k, \underline{r}) = \frac{4\pi}{c}\,\underline{I}\left(ka, \frac{\underline{r}}{a}\right) \tag{160}$$

In (160), the radius vector $\underline{r}$ on the surface of the disk of radius a is specified by the distance ρ from the center of the disk and the azimuthal angle ϕ measured from the positive x axis. According to (160), the electric current $\underline{J}$ in mksc units induced on the disk of radius a by $\underline{E}^{inc}$ of (147) is Bouwkamp's solution $\underline{I}$ for the disk of unit radius multiplied by $4\pi/c$ with Bouwkamp's k replaced by ka and with Bouwkamp's ρ replaced by ρ/a.

If

$$\underline{J} = \underline{u}_x J_x + \underline{u}_y J_y \tag{161}$$

then, thanks to the considerations in the previous paragraph and in the second paragraph of this section, Bouwkamp's formulas [3, Eqs. (23), (38), (39), Table I and Table II] yield

$$J_x = \frac{4(A + B\cos 2\phi)}{\pi\sqrt{1 - R^2}} \tag{162}$$

$$J_y = \frac{4B\sin 2\phi}{\pi\sqrt{1 - R^2}} \tag{163}$$

where

$$R = \rho/a \tag{164}$$

$$A = \sum_{n=1}^{\infty} A_n(-jka)^n \tag{165}$$

$$B = \sum_{n=1}^{\infty} B_n(-jka)^n \tag{166}$$

where the first six A_n's are given by

$$A_1 = \frac{-4 + 3R^2}{3} \tag{167a}$$

$$A_2 = 0 \tag{167b}$$

$$A_3 = \frac{56 - 40R^2 + 5R^4}{90} \tag{167c}$$

$$A_4 = \frac{4}{9\pi}(2 - R^2) \tag{167d}$$

$$A_5 = \frac{-2656 + 2408R^2 - 448R^4 + 21R^6}{12600} \tag{167e}$$

$$A_6 = \frac{2}{675\pi}(-296 + 192R^2 - 15R^4) \tag{167f}$$

and the first six B_n's are given by

$$B_1 = \frac{R^2}{3} \tag{168a}$$

$$B_2 = 0 \tag{168b}$$

$$B_3 = \frac{R^2(-8 + R^2)}{30} \tag{168c}$$

$$B_4 = -\frac{4R^2}{9\pi} \tag{168d}$$

$$B_5 = \frac{R^2(200 - 68R^2 + 3R^4)}{2520} \tag{168e}$$

$$B_6 = \frac{2R^2(134 - 15R^2)}{675\pi} \tag{168f}$$

As an alternative to (161) - (163), $\underline{J}$ can be written as

$$\underline{J} = \underline{u}_\rho J_\rho + \underline{u}_\phi J_\phi \tag{169}$$

where $\underline{u}_\rho$ and $\underline{u}_\phi$ are the unit vectors in the ρ and ϕ directions, respectively, and

$$J_\rho = \frac{4(A + B)\cos\phi}{\pi\sqrt{1 - R^2}} \tag{170}$$

$$J_\phi = \frac{4(B - A)\sin\phi}{\pi\sqrt{1 - R^2}} \tag{171}$$

It is evident from (165) and (166) that

$$A + B = \sum_{n=1}^{\infty} (A_n + B_n)(-jka)^n \tag{172}$$

$$B - A = \sum_{n=1}^{\infty} (B_n - A_n)(-jka)^n \tag{173}$$

Equations (167) and (168) give

$$A_1 + B_1 = -\frac{4}{3}(1 - R^2) \tag{174a}$$

$$A_2 + B_2 = 0 \tag{174b}$$

$$A_3 + B_3 = \frac{4}{45}(1 - R^2)(7 - R^2) \tag{174c}$$

$$A_4 + B_4 = \frac{8}{9\pi}(1 - R^2) \tag{174d}$$

$$A_5 + B_5 = \frac{(1-R^2)(-664 + 188R^2 - 9R^4)}{3150} \tag{174e}$$

$$A_6 + B_6 = \frac{4}{675\pi}(1 - R^2)(-148 + 15R^2) \tag{174f}$$

and

$$B_1 - A_1 = \frac{2}{3}(2 - R^2) \tag{175a}$$

$$B_2 - A_2 = 0 \tag{175b}$$

$$B_3 - A_3 = \frac{-28 + 8R^2 - R^4}{45} \tag{175c}$$

$$B_4 - A_4 = -\frac{8}{9\pi} \tag{175d}$$

$$B_5 - A_5 = \frac{1328 - 704R^2 + 54R^4 - 3R^6}{6300} \tag{175e}$$

$$B_6 - A_6 = \frac{4}{675\pi}(148 - 29R^2) \tag{175f}$$

For computation, (170) and (171) are rewritten as

$$J_\rho = C_\rho \cos\phi \tag{176}$$

$$J_\phi = C_\phi \sin\phi \tag{177}$$

where

$$C_\rho = \frac{4(A + B)}{\pi\sqrt{1 - R^2}} \tag{178}$$

$$C_\phi = \frac{4(B - A)}{\pi\sqrt{1 - R^2}} \tag{179}$$

In view of (174) and (175), substitution of the series (172) and (173), truncated at $n = 6$, into (178) and (179) gives

$$C_\rho = C_{\rho 1} + C_{\rho 2} + C_{\rho 3} + C_{\rho 4} + C_{\rho 5} \tag{180}$$

$$C_\phi = C_{\phi 1} + C_{\phi 2} + C_{\phi 3} + C_{\phi 4} + C_{\phi 5} \tag{181}$$

where

$$C_{\rho 1} = \frac{j4ka}{3} S_\rho \tag{182a}$$

$$C_{\rho 2} = \frac{j4(ka)^3(7 - R^2)}{45} S_\rho \tag{182b}$$

$$C_{\rho 3} = \frac{8(ka)^4}{9\pi} S_\rho \tag{182c}$$

$$C_{\rho 4} = \frac{-j(ka)^5(-664 + 188R^2 - 9R^4)}{3150} S_\rho \tag{182d}$$

$$C_{\rho 5} = -\frac{4(ka)^6(-148 + 15R^2)}{675\pi} S_\rho \tag{182e}$$

where

$$S_\rho = \frac{4\sqrt{1 - R^2}}{\pi} \tag{183}$$

Also,

$$C_{\phi 1} = \frac{j2ka(2 - R^2)}{3} S_\phi \tag{184a}$$

$$C_{\phi 2} = \frac{j(ka)^3(28 - 8R^2 + R^4)}{45} S_\phi \tag{184b}$$

$$C_{\phi 3} = \frac{8(ka)^4}{9\pi} S_\phi \tag{184c}$$

$$C_{\phi 4} = -\frac{j(ka)^5(-1328 + 704R^2 - 54R^4 + 3R^6)}{6300} S_\phi \tag{184d}$$

$$C_{\phi 5} = \frac{-4(ka)^6(-148 + 29R^2)}{675\pi} S_\phi \tag{184e}$$

where

$$S_\phi = -\frac{4}{\pi\sqrt{1 - R^2}} \tag{185}$$

Substitution of (176) and (177) into (169) gives

$$\underline{J} = \underline{u}_\rho C_\rho \cos\phi + \underline{u}_\phi C_\phi \sin\phi \tag{186}$$

Thus, the electric current $\underline{J}$ induced on the disk of radius a by the incident electric field $\underline{E}^{inc}$ of (147) is given by (186) where C_ρ and C_ϕ are given by (180) and (181), respectively.

The electric charge associated with $\underline{J}$ of (186) is called q_e and is given by the equation of continuity [1, Eq. (1)]. In view of [1, Eq. (B-3)], substitution of (186) into [1, Eq. (1)] gives

$$q_e = \frac{1}{-j\omega\rho} \left[\frac{\partial}{\partial\rho} (\rho C_\rho \cos \phi) + \frac{\partial}{\partial\phi} (C_\phi \sin \phi)\right] \tag{187}$$

which reduces to

$$q_e = \frac{C_q}{c} \cos \phi \tag{188}$$

where

$$C_q = \frac{j}{k\rho} \left(C_\rho + C_\phi + \rho \frac{\partial C_\rho}{\partial\rho}\right) \tag{189}$$

Thanks to (180)-(185), (189) becomes

$$C_q = C_{q1} + C_{q2} + C_{q3} + C_{q4} + C_{q5} \tag{190}$$

where

$$C_{q1} = 2 S_q \tag{191a}$$

$$C_{q2} = \frac{(ka)^2(4 - R^2)}{3} S_q \tag{191b}$$

$$C_{q3} = - \frac{j16(ka)^3}{9\pi} S_q \tag{191c}$$

$$C_{q4} = \frac{(ka)^4(88 - 44R^2 + 3R^4)}{180} S_q \tag{191d}$$

$$C_{q5} = \frac{j16(ka)^5(-26 + 5R^2)}{225\pi} S_q \tag{191e}$$

where

$$S_q = \frac{4R}{\pi\sqrt{1 - R^2}} \tag{192}$$

The electric current $\underline{J}$ induced on the disk is given in terms of C_ρ and C_ϕ by (186). The electric charge q_e induced on the disk is given in terms of C_q by (188). Now, C_ρ, C_ϕ, and C_q can be calculated by the computer program listed at the end of this section. This program reads input data from a punched card according to

```
          READ(1,12) N, BK
    12    FORMAT(I3, E14.7)
```

The coefficients C_ρ, C_ϕ, and C_q are calculated and printed out at

$$\rho/a = 0, \frac{1}{N}, \frac{2}{N}, \frac{3}{N}, \ldots \frac{N-1}{N}$$

where ρ is the distance from the center of the disk, a is the radius of the disk, and N is a positive integer. BK is the dimensionless product ka where k is the wave number.

The only array in the program is CQ. The minimum allocation for CQ is given by

```
          COMPLEX CQ(N)
```

The program prints out the real part, the imaginary part, and the magnitude of C_ρ of (180) under the heading "REAL JR IMAG JR MAG JR." The real part, the imaginary part, and the magnitude of C_ϕ of (181) are printed out under the heading "REAL JP IMAG JP MAG JP." The real part, the imaginary part, and the magnitude of C_q of (190) are printed out under the heading "REAL Q IMAG Q MAG Q." The ith row of numbers printed out under any one of these three headings is for

$$\rho/a = (i-1)/N$$

The sample input and output data are for the disk of radius 0.002 wavelengths with N = 30. The sample output data for $|C_\rho|$ are plotted as the solid curve in [1, Fig. 1]. Similarly, $|C_\phi|$ is plotted in [1, Fig. 2], and $|C_q|$ is plotted in [1, Fig. 3]. The solid curves were obtained by drawing straight line segments between the data at

$$\rho/a = 0, \quad 1/30, \quad 2/30, \quad 3/30, \; \ldots \; 29/30$$

It was possible to extend the curve for $|C_\rho|$ to the rim of the disk because $|C_\rho|$ is known to be zero there.

The function FABS(X) returns the magnitude of the complex variable X. If either of the real and imaginary parts of X is less than 10^{-35} in magnitude, then its contribution to the magnitude of X is omitted in order to avoid a machine underflow.

Inside DO loop 19 in the main program, C_ρ of (180) is accumulated in CR, C_ϕ of (181) is accumulated in CP, and C_q of (190) is accumulated in CQ(J). The index J of DO loop 19 obtains

$$\rho/a = (J-1)/N$$

In accordance with (164), line 28 puts ρ/a in R. Line 31 puts S_ρ of (183) in SR, line 32 puts S_ϕ of (185) in SP, and line 33 puts S_q of (192) in SQ. Line 34 puts $C_{\rho 1}$ of (182a) in CR, line 35 puts $C_{\phi 1}$ of (184a) in CP, and line 36 puts C_{q1} of (191a) in CQ(J). If ka < 10^{-20}, nothing is added to CR, CP, and CQ(J) for fear of machine underflows. If ka $\geq 10^{-20}$, lines 40-42 add $C_{\rho 2}$, $C_{\phi 2}$, and C_{q2} to CR, CP, and CQ(J), respectively. If ka $\geq 10^{-15}$, lines 45-47 add $C_{\rho 3}$, $C_{\phi 3}$, and C_{q3} to

CR, CP, and CQ(J). If ka $\geq 10^{-12}$, lines 50-52 add $C_{\rho 4}$, $C_{\phi 4}$, and C_{q4} to CR, CP, and CQ(J). If ka $\geq 10^{-10}$, lines 55-57 add $C_{\rho 5}$, $C_{\phi 5}$, and C_{q5} to CR, CP, and CQ(J). Line 58 puts the magnitude of CR in SR, and line 59 puts the magnitude of CP in SP.

DO loop 17 prints out CQ(J).

```
C     LISTING OF THE COMPUTER PROGRAM THAT CALCULATES BOUWKAMP'S
C     POWER SERIES SOLUTION FOR A CIRCULAR DISK
//PGM JOB (XXXX,XXXX,1,1),'MAUTZ,JOE',REGION=200K
// EXEC WATFIV
//GO.SYSIN DD *
$JOB              MAUTZ,TIME=1,PAGES=40
      FUNCTION FABS(X)
      COMPLEX X
      R=ABS(REAL(X))
      A=ABS(AIMAG(X))
      IF(R.LT.1.E-35) R=0.
      IF(A.LT.1.E-35) A=0.
      FABS=SQRT(R*R+A*A)
      RETURN
      END
      COMPLEX U,CR,CP,CQ(180)
      C=4./3.141593
      U=(0.,1.)
      READ(1,12) N,BK
   12 FORMAT(I3,E14.7)
      WRITE(3,14) N,BK
   14 FORMAT('0  N',6X,'BK'/1X,I3,E14.7)
      D=1./N
      WRITE(3,13)
   13 FORMAT('0  REAL JR    IMAG JR     MAG JR',6X,'REAL JP    IMAG JP
     1    MAG JP')
      DO 19 J=1,N
      R=(J-1)*D
      R2=R*R
      S=SQRT(1.-R2)
      SR=C*S
      SP=-C/S
      SQ=-SP*R
      CR=1.333333*SR*BK*U
      CP=.6666667*SP*(2.-R2)*BK*U
      CQ(J)=SQ*2.
      IF(BK.LT.1.E-20) GO TO 11
      BK2=BK*BK
      BK3=BK2*BK
      CR=.8888889E-01*SR*(7.-R2)*BK3*U+CR
      CP=.2222222E-01*SP*(R2*(R2-8.)+28.)*BK3*U+CP
      CQ(J)=SQ*BK2*(4.-R2)/3.+CQ(J)
      IF(BK.LT.1.E-15) GO TO 11
      BK4=BK3*BK
      CR=.2829421*SR*BK4+CR
      CP=.2829421*SP*BK4+CP
      CQ(J)=-.5658842*SQ*BK3*U+CQ(J)
      IF(BK.LT.1.E-12) GO TO 11
      BK5=BK4*BK
      CR=-1./3150.*SP*((188.-9.*R2)*R2-664.)*BK5*U+CR
      CP=-1./6300.*SP*(((3.*R2-54.)*R2+704.)*R2-1328.)*BK5*U+CP
      CQ(J)=SQ*BK4*(R2*(3.*R2-44.)+88.)/180.+CQ(J)
      IF(BK.LT.1.E-10) GO TO 11
      BK6=BK5*BK
      CR=-.1886281E-02*SR*(15.*R2-148.)*BK6+CR
      CP=-.1886281E-02*SP*(29.*R2-148.)*BK6+CP
      CQ(J)=SQ*BK5*.2263537E-01*(-26.+5.*R2)*U+CQ(J)
   11 SR=FABS(CR)
      SP=FABS(CP)
      WRITE(3,15) CR,SR,CP,SP
```

```
   15 FORMAT(1X,3E11.4,2X,3E11.4)
   19 CONTINUE
      WRITE(3,18)
   18 FORMAT('0   REAL Q      IMAG Q      MAG Q')
      DO 17 J=1,N
      SQ=FABS(CQ(J))
      WRITE(3,15) CQ(J),SQ
   17 CONTINUE
      STOP
      END
$DATA
 30 0.1256637E-01
$STOP
/*
//
```

PRINTED OUTPUT

N	BK
30	0.1256637E-01

REAL JR	IMAG JR	MAG JR	REAL JP	IMAG JP	MAG JP
0.8985E-08	0.2133E-01	0.2133E-01	-0.8985E-08	-0.2133E-01	0.2133E-01
0.8980E-08	0.2132E-01	0.2132E-01	-0.8990E-08	-0.2133E-01	0.2133E-01
0.8965E-08	0.2129E-01	0.2129E-01	-0.9005E-08	-0.2133E-01	0.2133E-01
0.8940E-08	0.2123E-01	0.2123E-01	-0.9030E-08	-0.2134E-01	0.2134E-01
0.8905E-08	0.2114E-01	0.2114E-01	-0.9066E-08	-0.2134E-01	0.2134E-01
0.8859E-08	0.2104E-01	0.2104E-01	-0.9112E-08	-0.2134E-01	0.2134E-01
0.8803E-08	0.2090E-01	0.2090E-01	-0.9170E-08	-0.2134E-01	0.2134E-01
0.8737E-08	0.2075E-01	0.2075E-01	-0.9240E-08	-0.2134E-01	0.2134E-01
0.8660E-08	0.2056E-01	0.2056E-01	-0.9322E-08	-0.2135E-01	0.2135E-01
0.8571E-08	0.2035E-01	0.2035E-01	-0.9419E-08	-0.2136E-01	0.2136E-01
0.8471E-08	0.2011E-01	0.2011E-01	-0.9530E-08	-0.2137E-01	0.2137E-01
0.8359E-08	0.1985E-01	0.1985E-01	-0.9657E-08	-0.2139E-01	0.2139E-01
0.8235E-08	0.1955E-01	0.1955E-01	-0.9803E-08	-0.2142E-01	0.2142E-01
0.8097E-08	0.1923E-01	0.1923E-01	-0.9970E-08	-0.2145E-01	0.2145E-01
0.7947E-08	0.1887E-01	0.1887E-01	-0.1016E-07	-0.2150E-01	0.2150E-01
0.7781E-08	0.1848E-01	0.1848E-01	-0.1037E-07	-0.2156E-01	0.2156E-01
0.7600E-08	0.1805E-01	0.1805E-01	-0.1062E-07	-0.2163E-01	0.2163E-01
0.7403E-08	0.1758E-01	0.1758E-01	-0.1090E-07	-0.2174E-01	0.2174E-01
0.7188E-08	0.1707E-01	0.1707E-01	-0.1123E-07	-0.2187E-01	0.2187E-01
0.6953E-08	0.1651E-01	0.1651E-01	-0.1161E-07	-0.2204E-01	0.2204E-01
0.6697E-08	0.1590E-01	0.1590E-01	-0.1205E-07	-0.2226E-01	0.2226E-01
0.6416E-08	0.1524E-01	0.1524E-01	-0.1258E-07	-0.2256E-01	0.2256E-01
0.6109E-08	0.1450E-01	0.1450E-01	-0.1322E-07	-0.2294E-01	0.2294E-01
0.5769E-08	0.1370E-01	0.1370E-01	-0.1399E-07	-0.2346E-01	0.2346E-01
0.5391E-08	0.1280E-01	0.1280E-01	-0.1497E-07	-0.2418E-01	0.2418E-01
0.4967E-08	0.1179E-01	0.1179E-01	-0.1625E-07	-0.2520E-01	0.2520E-01
0.4482E-08	0.1064E-01	0.1064E-01	-0.1801E-07	-0.2670E-01	0.2670E-01
0.3916E-08	0.9300E-02	0.9300E-02	-0.2061E-07	-0.2912E-01	0.2912E-01
0.3226E-08	0.7659E-02	0.7659E-02	-0.2503E-07	-0.3354E-01	0.3354E-01
0.2300E-08	0.5462E-02	0.5462E-02	-0.3509E-07	-0.4440E-01	0.4440E-01

REAL Q	IMAG Q	MAG Q
0.0000E+00	0.0000E+00	0.0000E+00
0.8494E-01	-0.4769E-07	0.8494E-01
0.1702E+00	-0.9555E-07	0.1702E+00
0.2560E+00	-0.1437E-06	0.2560E+00
0.3426E+00	-0.1924E-06	0.3426E+00
0.4305E+00	-0.2417E-06	0.4305E+00
0.5199E+00	-0.2919E-06	0.5199E+00

```
0.6111E+00-0.3431E-06 C.6111E+00
0.7046E+00-0.3957E-06 0.7046E+00
0.8009E+00-0.4497E-06 0.8009E+00
0.9004E+00-0.5056E-06 0.9004E+00
0.1004E+01-0.5636E-06 0.1004E+01
0.1111E+01-0.6241E-06 0.1111E+01
0.1225E+01-0.6876E-06 0.1225E+01
0.1344E+01-C.7545E-06 0.1344E+01
0.1470E+01-0.8256E-06 0.1470E+01
0.1606E+01-0.9016E-06 0.1606E+01
0.1751E+01-0.9835E-06 0.1751E+01
0.1910E+01-0.1072E-05 0.1910E+01
0.2084E+01-0.1170E-05 0.2084E+01
0.2278E+01-0.1279E-05 0.2278E+01
0.2496E+01-0.1402E-05 0.2496E+01
0.2747E+01-0.1542E-05 0.2747E+01
0.3041E+01-0.1708E-05 0.3041E+01
0.3396E+01-0.1907E-05 0.3396E+01
0.3839E+01-0.2156E-05 0.3839E+01
0.4424E+C1-0.2484E-05 0.4424E+01
0.5258E+01-0.2953E-05 C.5258E+01
0.6621E+01-0.3718E-05 0.6621E+01
0.9615E+01-0.5399E-05 0.9615E+01
```

VIII. THE MIE SERIES SOLUTION FOR A SPHERE

The known solutions for the electric current and electric charge on a conducting sphere excited by an incident plane wave are plotted in [1, Figs. 4, 5, and 6]. The known solution for the electric current was obtained from the Mie series solution [4, Eq. (6-103)]. Actually, [4, Eq. (6-103)] had to be modified so as to be valid for the incident plane wave [1, Eq. (110)] which travels in the negative z direction rather than for the incident plane wave [4, Eq. (6-96)] which travels in the positive z direction. The electric charge was obtained via the equation of continuity. The foregoing electric current and electric charge were calculated by means of a computer program. This program is listed at the end of this section.

If E_o is set equal to η, if $P_n^1(\cos\theta)$ is replaced by $-P_n^1(\cos\theta)$, and if $P_n^{1'}(\cos\theta)$ is replaced by $-P_n^{1'}(\cos\theta)$, then [4, Eq. (6-103)] becomes

$$\underline{\hat{J}}(\theta,\phi) = \underline{u}_\theta \hat{J}_\theta(\theta)\cos\phi + \underline{u}_\phi \hat{J}_\phi(\theta)\sin\phi \tag{193}$$

where

$$\hat{J}_\theta(\theta) = \frac{1}{ka}\sum_{n=1}^{\infty} a_n \left(\frac{j\frac{d}{d\theta}P_n^1(\cos\theta)}{\hat{H}_n^{(2)'}(ka)} + \frac{P_n^1(\cos\theta)}{\sin\theta\,\hat{H}_n^{(2)}(ka)}\right) \tag{194}$$

$$\hat{J}_\phi(\theta) = \frac{1}{ka}\sum_{n=1}^{\infty} a_n \left(\frac{-jP_n^1(\cos\theta)}{\sin\theta\,\hat{H}_n^{(2)'}(ka)} - \frac{\frac{d}{d\theta}P_n^1(\cos\theta)}{\hat{H}_n^{(2)}(ka)}\right) \tag{195}$$

where

$$a_n = \frac{j^{-n}(2n+1)}{n(n+1)} \tag{196}$$

In (193), $\underline{u}_\theta$ and $\underline{u}_\phi$ are the unit vectors in the θ and ϕ directions, respectively. As given by (193), $\underline{\hat{J}}(\theta,\phi)$ is the electric current on the conducting

sphere of radius a centered at the origin and excited by the plane wave whose electric field is $\hat{\underline{E}}^{inc}$ given by

$$\hat{\underline{E}}^{inc} = \underline{u}_x \eta \, e^{-jkz} \tag{197}$$

It was necessary to replace P_n^1 by $-P_n^1$ and $P_n^{1'}$ by $-P_n^{1'}$ in [4, Eq. (6-103)] because the P_n^1 used in the present report is taken to be the negative of the P_n^1 used in [4]. The P_n^1 used in the present report is that which is calculated in [5]. The right-hand sides of (194) and (195) coincide with the right-hand sides of [5, Eqs. (2) and (3)].

The electric charge associated with the electric current (193) is called $\hat{q}_e(\theta,\phi)$. As calculated from [1, Eq. (1)] with $\nabla_s \cdot$ given by [1, Eq. (B-3)],

$$\hat{q}_e(\theta,\phi) = \frac{\hat{q}(\theta)}{c} \cos\phi \tag{198}$$

where

$$\hat{q}(\theta) = \frac{j}{ka \sin\theta} \left(\frac{d}{d\theta} (\sin\theta \, \hat{J}_\theta(\theta)) + \hat{J}_\phi(\theta) \right) \tag{199}$$

Substitution of (194) and (195) into (199) produces

$$\hat{q}(\theta) = \frac{1}{(ka)^2} \sum_{n=1}^{\infty} \frac{a_n}{\hat{H}_n^{(2)'}(ka)} \left(-\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} P_n^1(\cos\theta)\right) + \frac{P_n^1(\cos\theta)}{\sin^2\theta} \right) \tag{200}$$

Thanks to the associated Legendre equation [4, Eq. (E-1)]

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} P_n^1(\cos\theta)\right) + \left[n(n+1) - \frac{1}{\sin^2\theta}\right] P_n^1(\cos\theta) = 0 \tag{201}$$

expression (200) reduces to

$$\hat{q}(\theta) = \frac{1}{(ka)^2} \sum_{n=1}^{\infty} \frac{a_n n(n+1) P_n^1(\cos\theta)}{\hat{H}_n^{(2)'}(ka)} \tag{202}$$

So far, it has been established that the incident plane wave (197), which travels in the positive z direction, induces the electric current (193) and the electric charge (198) on the sphere. However, the objective is to find the electric current and the electric charge induced on the sphere by an incident plane wave that travels in the negative z direction.

If the situation in which the electric current (193) exists on the sphere excited by the incident electric field (197) is rotated 180° about the x axis, then the resulting situation is that of the electric current induced on the sphere excited by the incident plane wave whose electric field is $\underline{E}^{inc}$ given by

$$\underline{E}^{inc} = \underline{u}_x \eta \, e^{jkz} \tag{203}$$

This plane wave travels in the negative z direction and coincides with [1, Eq. (110)]. Therefore, the electric current $\underline{J}$ induced on the sphere in [1] is $\hat{\underline{J}}$ of (193) rotated 180° about the x axis.

Substitution of

$$\underline{u}_\theta = \underline{u}_x \cos\theta \cos\phi + \underline{u}_y \cos\theta \sin\phi - \underline{u}_z \sin\theta \tag{204}$$

$$\underline{u}_\phi = -\underline{u}_x \sin\phi + \underline{u}_y \cos\phi \tag{205}$$

into (193) gives

$$\hat{\underline{J}}(\theta,\phi) = (\underline{u}_x \cos\theta \cos\phi + \underline{u}_y \cos\theta \sin\phi - \underline{u}_z \sin\theta)\, \hat{J}_\theta(\theta) \cos\phi$$

$$+ (-\underline{u}_x \sin\phi + \underline{u}_y \cos\phi)\, \hat{J}_\phi(\theta) \sin\phi \tag{206}$$

Expression (206) is more suitable than (193) for 180° rotation about the x axis. A 180° rotation about the x axis amounts to replacement of x by itself, replacement of y by -y, and replacement of z by -z. This is written as

$$\left.\begin{array}{l} x \to x \\ y \to -y \\ z \to -z \end{array}\right\} \tag{207}$$

Now, (207) is equivalent to

$$\left.\begin{array}{l} \theta \to \pi - \theta \\ \phi \to -\phi \end{array}\right\} \tag{208}$$

When rotated 180° about the x axis, $\underline{u}_x$ remains unchanged, $\underline{u}_y$ changes to $-\underline{u}_y$, and $\underline{u}_z$ changes to $-\underline{u}_z$. It is now evident that $\hat{\underline{J}}(\theta,\phi)$ of (206) can be rotated 180° about the x axis by taking the value of $\hat{\underline{J}}(\theta,\phi)$ with the signs of its y and z components changed and placing this value not at (θ,ϕ) but at $(\pi-\theta, -\phi)$. Hence, if the result of 180° rotation of $\hat{\underline{J}}(\theta,\phi)$ about the x axis is called $\underline{J}(\theta,\phi)$, then

$$\underline{J}(\pi-\theta, -\phi) = (\underline{u}_x \cos\theta\cos\phi - \underline{u}_y \cos\theta\sin\phi + \underline{u}_z \sin\theta)\, \hat{J}_\theta(\theta)\cos\phi$$
$$+ (-\underline{u}_x \sin\phi - \underline{u}_y \cos\phi)\, \hat{J}_\phi(\theta)\sin\phi \tag{209}$$

Replacement of θ by $\pi-\theta$ and ϕ by $-\phi$ in the arguments of $\underline{J}$, $\hat{J}_\theta$, $\hat{J}_\phi$, the sines, and the cosines in (209) yields

$$\underline{J}(\theta,\phi) = -(\underline{u}_x \cos\theta\cos\phi + \underline{u}_y \cos\theta\sin\phi - \underline{u}_z \sin\theta)\, \hat{J}_\theta(\pi-\theta)\cos\phi$$
$$+ (-\underline{u}_x \sin\phi + \underline{u}_y \cos\phi)\, \hat{J}_\phi(\pi-\theta)\sin\phi \tag{210}$$

Now, (204) and (205) reduce (210) to

$$\underline{J}(\theta,\phi) = -\underline{u}_\theta \hat{J}_\theta(\pi-\theta)\cos\phi + \underline{u}_\phi \hat{J}_\phi(\pi-\theta)\sin\phi \tag{211}$$

The electric charge associated with the electric current (211) is called $q_e(\theta,\phi)$. As calculated from [1, Eq. (1)] with $\nabla_s\cdot$ given by [1, Eq. (B-3)],

$$q_e(\theta,\phi) = \frac{\hat{q}(\pi-\theta)}{c} \cos \phi \tag{212}$$

where $\hat{q}(\theta)$ is given by (199) which was previously reduced to (202).

According to the development (203)-(212), the incident plane wave (203), which travels in the negative z direction, induces the electric current (211) and the electric charge (212) on the sphere. The computer program listed at the end of this section calculates the functions $\hat{J}_\theta(\theta)$, $\hat{J}_\phi(\theta)$, and $\hat{q}(\theta)$ that appear in (211) and (212). Of course, these functions also appear in expressions (193) and (198) for the electric current and electric charge induced on the sphere by the incident plane wave (197), which travels in the positive z direction.

The computer program listed at the end of this section contains the subroutines BES, LEG and CURNTC. The subroutines BES and LEG are exactly the same as in [5] and are described in detail there.

The subroutine CURNTC(X, NT, CUR, Q) puts $\hat{J}_\theta(\theta)$ of (194) for $\theta = (K-1)\pi/(NT-1)$ in CUR(K) for K=1,2,...NT. Similarly, CURNTC puts $\hat{J}_\phi(\theta)$ of (195) in CUR(NT+1) through CUR(2*NT) and $\hat{q}(\theta)$ of (202) in Q(1) through Q(NT). The value of ka in (194), (195), and (202) is equal to X. The arguments X and NT are input arguments and the arguments CUR and Q are output arguments. The subroutine CURNTC calls the subroutines BES and LEG. Minimum allocations in the subroutine CURNTC are given by

```
COMPLEX CUR(2*NT), Q(NT), H(N), HP(N), HQ(N)
DIMENSION BJ(2*N+1), BJP(2*N+1), BY(N+1), BYP(N+1)
```

where

$$N = MIN(5.+4.*X, 29.) \tag{213}$$

Here, MIN denotes minimum value.

The subroutine CURNTC(X, NT, CUR, Q) was created by appending Q to the list of arguments of the subroutine CURNTC(X, NT, CUR) listed in [5, p. 11] and calculating Q by means of (202). For the calculations, the $\sum_{n=1}^{\infty}$ in (194), (195), and (202) was replaced by $\sum_{n=1}^{N}$ where N is given by (213).

In the subroutine CURNTC, statement 14 puts the spherical Bessel function of the first kind $j_n(ka)$ in BJ(n+1), $j_n'(ka)$ in BJP(n+1), the spherical Bessel function of the second kind $y_n(ka)$ in BY(n+1), and $y_n'(ka)$ in BYP(n+1) for n = 0, 1,2,...N. The functions $\hat{H}_n^{(2)}(ka)$ and $\hat{H}_n^{(2)'}(ka)$ in (194), (195), and (202) are given in terms of the spherical Bessel functions and their derivatives by

$$\hat{H}_n^{(2)}(ka) = ka(j_n(ka) - jy(ka)) \tag{214}$$

$$\hat{H}_n^{(2)'}(ka) = kaj_n'(ka) + j_n(ka) - j(ka\, y_n'(ka) + y_n(ka)) \tag{215}$$

DO loop 11 puts $\dfrac{a_n}{ka\ \hat{H}_n^{(2)}(ka)}$ in H(J) for n = J. Furthermore, DO loop 11 puts $\dfrac{ja_n}{ka\ \hat{H}_n^{(2)'}(ka)}$ in HP(J) and $\dfrac{a_n n(n+1)}{(ka)^2\ \hat{H}_n^{(2)'}(ka)}$ in HQ(J). The index K of DO loop 12 obtains θ according to $\theta = (K-1)\pi/(NT-1)$. Statement 15 puts $P_n^1(\cos\theta)$ in BJ(N+1+n) and $\frac{d}{d\theta} P_n^1(\cos\theta)$ in BJP(N+1+n) for n = 1,2,...N. DO loop 13 accumulates $\hat{J}_\theta(\theta)$ of (194), $\hat{J}_\phi(\theta)$ of (195), and $\hat{q}(\theta)$ of (202) in G1, G2, and G3, respectively.

The main program reads input data from a punched card according to

```
      READ(1,23) NT, XC
23    FORMAT(I3, E14.7)
```

For ka = XC, the main program prints out the quantities $\hat{J}_\theta(\theta)$ of (194), $\hat{J}_\phi(\theta)$ of (195), and $\hat{q}(\theta)$ of (202) at

$$\theta = (i-1)\pi/(NT-1), \quad i = 1,2,...NT \tag{216}$$

where NT is a positive integer and NT > 1. The main program requires the subroutines CURNTC, BES, and LEG. Minimum allocations in the main program are given by

```
COMPLEX CUR(2*NT), Q(NT)
```

The main program prints out the real part and the imaginary part of $\hat{J}_\theta(\theta)$ under the heading "REAL JT IMAG JT." The real and imaginary parts of $\hat{J}_\phi(\theta)$ are printed out under the heading "REAL JP IMAG JP." The magnitude of $\hat{J}_\theta(\theta)$ is printed out under the heading "MAG JT." The magnitude of $\hat{J}_\phi(\theta)$ is printed out under the heading "MAG JP." Down farther, the real part, the imaginary part, and the magnitude of $\hat{q}(\theta)$ are printed out under the heading "REAL Q IMAG Q MAG Q." The ith row of numbers printed out under any one of these headings is for θ given by (216). The sample input and output data are for the sphere of radius 0.002 wavelengths with NT = 31. The sample output data for $|\hat{J}_\theta(\theta)|$ are plotted as the solid curve in [1, Fig. 4]. The solid curve was obtained by drawing straight line segments between the data at the values of θ given by (216). Actually, $|\hat{J}_\theta(0)|$ was plotted at t = 0 in [1, Fig. 4], and $|\hat{J}_\theta(\pi)|$ was plotted at t = πa in [1, Fig. 4]. However t = 0 corresponds to $\theta = \pi$, and t = πa corresponds to $\theta = 0$ so that the solid curve in [1, Fig. 4] is really a plot of $|\hat{J}_\theta(\pi-\theta)|$. This is in harmony with expression (211) for the electric current on the sphere excited, as in [1, Fig. 4], by the incident plane wave (203), which travels in the negative z direction. Similarly,

the sample output data for $|\hat{J}_\phi(\pi-\theta)|$ are plotted as the solid curve in [1, Fig. 5], and the sample output data for $|\hat{q}(\pi-\theta)|$ are plotted as the solid curve in [1, Fig. 6].

If $ka \leq 0.001$, then the subroutine CURNTC can not be used to calculate $\hat{J}_\theta(\theta)$, $\hat{J}_\phi(\theta)$, and $\hat{q}(\theta)$ because the subroutine BES called by it does not calculate any of the functions $j_n'(ka)$, $y_n(ka)$ and $y_n'(ka)$ whenever $ka \leq 0.001$. If ka is small, then the spherical Bessel functions $j_n(ka)$ and $y_n(ka)$ can be approximated by [6, Eqs. (10.1.4) and (10.1.5)]

$$j_n(ka) = \frac{(ka)^n}{1\cdot 3\cdot 5\ldots(2n+1)} \tag{217}$$

$$y_n(ka) = \frac{-1\cdot 3\cdot 5\ldots(2n-1)}{(ka)^{n+1}} \tag{218}$$

Therefore, when ka is small, $\hat{H}_n^{(2)}(ka)$ of (214) and $\hat{H}_n^{(2)\prime}(ka)$ of (215) can be approximated by

$$\hat{H}_n^{(2)}(ka) = \frac{j(1\cdot 3\cdot 5\ldots(2n-1))}{(ka)^n} \tag{219}$$

$$\hat{H}_n^{(2)\prime}(ka) = \frac{-jn(1\cdot 3\cdot 5\ldots(2n-1))}{(ka)^{n+1}} \tag{220}$$

When ka is small, it suffices to retain only the term for which $n = 1$ in each of the infinite series on the right-hand sides of (194), (195), and (202). The required $P_1^1(\cos\theta)$, being the negative of the $P_1^1(\cos\theta)$ defined by [4, Eq. (E-17)], is given by

$$P_1^1(\cos\theta) = \sqrt{1-\cos^2\theta} = \sin \tag{221}$$

In view of (219), (220), and (221), retention of only the term for which $n = 1$ in each of the infinite series on the right-hand sides of (194), (195), and (202) gives

$$\hat{J}_\theta(\theta) = -1.5 \quad (222)$$

$$\hat{J}_\phi(\theta) = 1.5 \cos \theta \quad (223)$$

$$\hat{q}(\theta) = 3 \sin \theta \quad (224)$$

If ka $\leq$ 0.001, then the branch statement on line 198 in the main program causes DO loop 16 to be executed. The index J of DO loop 16 obtains $\theta = (J-1)\pi/(NT-1)$. Inside DO loop 16, $\hat{J}_\theta(\theta)$ of (222) is put in CUR(J), $\hat{J}_\phi(\theta)$ of (223) is put in CUR(J+NT), and $\hat{q}(\theta)$ of (224) is put in Q(J).

If ka > 0.001, then the branch statement on line 198 causes statement 15 to be executed. For the NT values of θ given by (216), statement 15 puts $\hat{J}_\theta(\theta)$ of (194) in CUR(1) through CUR(NT), $\hat{J}_\phi(\theta)$ of (195) in CUR(NT+1) through CUR(2*NT), and $\hat{q}(\theta)$ of (202) in Q(1) through Q(NT). DO loop 10 prints out $\hat{J}_\theta(\theta)$ and $\hat{J}_\phi(\theta)$. DO loop 13 prints out $\hat{q}(\theta)$.

```
C       LISTING OF THE COMPUTER PROGRAM FOR THE MIE SERIES SOLUTION
C       FOR A SPHERE
//PGM JOB (XXXX,XXXX,1,1),'MAUTZ.JCE',REGION=200K
// EXEC WATFIV
//GO.SYSIN DD *
$JOB               MAUTZ,TIME=1,PAGES=40
C
C       LISTING OF THE SUBROUTINE BES
        SUBROUTINE BES(L,LD,ID,NJ,XJ,BJ,BJP,BY,BYP)
        DIMENSION BJ(50),BJP(50),BY(50),BYP(50),BS(50)
        L1=(L-1)*NJ
        L3=(LD-1)*NJ
      6 IF(XJ-1.E-3) 3,3,4
      3 J1=L1+1
        J2=L1+NJ
        DO 5 J=J1,J2
        BJ(J)=0.
      5 CONTINUE
        BJ(J1)=1.
        RETURN
      4 SN=SIN(XJ)
        CS=COS(XJ)
        IF(XJ-15.) 11,12,12
     12 BJ(L1+1)=SN/XJ
        BJ(L1+2)=(BJ(L1+1)-CS)/XJ
        DO 14 I=3,NJ
        I3=L1+I
        I2=I3-1
        I1=I3-2
        BJ(I3)=FLOAT(2*I-3)/XJ*BJ(I2)-BJ(I1)
     14 CONTINUE
        B3=FLOAT(2*NJ-1)/XJ*BJ(I3)-BJ(I2)
        GO TO 15
     11 NBJ=XJ+22.
        IF(NBJ.LE.NJ) NBJ=NJ+1
        BS(NBJ)=0.
        BS(NBJ-1)=1.E-50
        NBJ2=NBJ-2
        DO 193 I=1,NBJ2
        I2=NBJ-I
        I3=I2+1
        I1=I2-1
        FI=FLOAT(2*I1+1)/XJ
        BS(I1)=BS(I2)*FI-BS(I3)
    193 CONTINUE
        B1=SN/XJ
        B2=B1/XJ-CS/XJ
        IF(ABS(B1)-ABS(B2))1,2,2
      2 BB=B1/BS(1)
        GO TO 9
      1 BB=B2/BS(2)
      9 DO 194 I=1,NJ
        I1=L1+I
        BJ(I1)=BS(I)*BB
    194 CONTINUE
        B3=BS(NJ+1)*BB
     15 BY(L1+1)=-CS/XJ
        BY(L1+2)=(BY(L1+1)-SN)/XJ
        DO 64 I=3,NJ
        I3=L1+I
```

```
2=I3-1
1=I3-2
Y(I3)=FLOAT(2*I-3)/XJ*BY(I2)-BY(I1)
ONTINUE
4=FLOAT(2*NJ-1)/XJ*BY(I3)-BY(I2)
F(ID.EQ.2) RETURN
J1=NJ-1
1=L3+1
2=L1+2
JP(J1)=-BJ(J2)
YP(J1)=-BY(J2)
) 65 J=2,NJ1
2=L3+J
1=L1+J-1
3=J1+2
J=2*(2*J-1)
JP(J2)=.5*(BJ(J1)-BJ(J3))-(BJ(J1)+BJ(J3))/FJ
YP(J2)=.5*(BY(J1)-BY(J3))-(BY(J1)+BY(J3))/FJ
ONTINUE
J=FJ+4.
2=J2+1
1=J1+1
JP(J2)=.5*(BJ(J1)-B3)-(BJ(J1)+B3)/FJ
YP(J2)=.5*(BY(J1)-B4)-(BY(J1)+B4)/FJ
ETURN
ND

ISTING OF THE SUBROUTINE LEG
JBROUTINE LEG(L,LO,ID,NJ,M,XP,P,PP)
IMENSION PC(8),P(59),PP(59),PS(50)
C(1)=1.
1=M+1
) 7 J=1,M1
C(J+1)=PC(J)*FLOAT(2*J-1)
ONTINUE
5=M*NJ-M*(M-1)/2
2=(L-1)*L5
)=(LC-1)*L5
2=ABS(1.-XP*XP)
.=SQRT(X2)
) 3 J=1,M1
2=L2+(J-1)*NJ-(J-2)*(J-1)/2
3=1.
(J.NE.1) X3=X1**(J-1)
S(1)=PC(J)*X3
S(2)=PC(J+1)*XP*X3
(J.EQ.M1) GO TC 14
M2+1)=PS(1)
M2+2)=PS(2)
J1=NJ-J+1
 4 I=3,NJ1
=I-2
=I-1
(I)=2.*XP*PS(I2)-PS(I1)+FLOAT(2*J-3)/FLOAT(I2)*(XP*PS(I2)-PS(I1))

(J.EQ.M1) GO TC 4
=M2+I
J2)=PS(I)
NTINUE
NTINUE
```

```
      IF(ID.EQ.2) RETURN
      DO 5 J=1,M
      M2=L4+(J-1)*NJ-(J-2)*(J-1)/2
      M3=M2+L2-L4
      NJ1=NJ-J+1
      DO 6 I=2,NJ1
      J2=M2+I
      J1=M3+I-NJ+J-1
      J3=M3+I+NJ-J
      IF(J.NE.1.AND.J.NE.M) GO TO 8
      IF(J.NE.1) GO TO 12
      PP(J2)=-P(J3)
      GO TO 6
   12 PP(J2)=.5*(FLOAT(I*(2*J+I-3))*P(J1)-PS(I-1))
      GO TO 6
    8 PP(J2)=.5*(FLOAT(I*(2*J+I-3))*P(J1)-P(J3))
    6 CONTINUE
      J2=M2+1
      J1=M3-NJ+J
      IF(J.NE.1) GO TO 13
      PP(J2)=0.
      GO TO 5
   13 PP(J2)=.5*FLOAT(2*J-2)*P(J1)
    5 CONTINUE
      RETURN
      END
C
C     LISTING OF THE SUBROUTINE CURNTC
C     THE SUBROUTINE CURNTC CALLS THE SUBROUTINES BES AND LEG
      SUBROUTINE CURNTC(X,NT,CUR,Q)
      COMPLEX CUR(146),Q(73),U,G1,G2,G3,G4,H(29),HP(29),HQ(29)
      DIMENSION BJ(59),BJP(59),BY(50),BYP(50)
      N=MIN1(5.+4.*X,29.)
      NP=N+1
   14 CALL BES(1,1,1,NP,X,BJ,BJP,BY,BYP)
      U=(0.,1.)
      G1=1.
      DO 11 J=1,N
      G1=-G1*U
      G3=FLOAT(2*J+1)/X*G1
      J1=J+1
      G2=1./FLOAT(J*J1)*G3
      G4=1./(X*BJP(J1)+BJ(J1)-U*(X*BYP(J1)+BY(J1)))
      H(J)=G2/(X*(BJ(J1)-U*BY(J1)))
      HP(J)=G4*G2*U
      HQ(J)=G4/X*G3
   11 CONTINUE
      DT=3.141593/(NT-1)
      DO 12 K=1,NT
      Z=COS((K-1)*DT)
   16 IF(ABS(Z).GE.1.) Z=SIGN(.99998,Z)
      SN=1./SQRT(1.-Z*Z)
   15 CALL LEG(1,1,1,NP,2,Z,BJ,BJP)
      G1=0.
      G2=0.
      G3=0.
      DO 13 J=1,N
      J1=NP+J
      SB=SN*BJ(J1)
      G1=G1+BJP(J1)*HP(J)+SB*H(J)
```

```
      G2=G2-SB*HP(J)-BJP(J1)*H(J)
      G3=G3+HC(J)*EJ(J1)
   13 CONTINUE
      CUR(K)=G1
      CUR(K+NT)=G2
      Q(K)=G3
   12 CONTINUE
      RETURN
      END
C
C     LISTING OF THE MAIN PROGRAM
C     THE SUBROUTINES CURNTC, BES, AND LEG ARE REQUIRED
      COMPLEX CUR(292),Q(146),C2
      READ(1,23) NT,XC
   23 FORMAT(I3,E14.7)
      WRITE(3,24) NT,XC
   24 FORMAT('  NT',6X,'XC'/1X,I3,E14.7)
      IF(XC-1.E-3) 14,14,15
   14 DT=3.141593/(NT-1)
      DO 16 J=1,NT
      TH=(J-1)*DT
      CUR(J)=-1.5
      CUR(J+NT)=1.5*COS(TH)
      Q(J)=3.*SIN(TH)
   16 CONTINUE
      GO TO 17
   15 CALL CURNTC(XC,NT,CUR,Q)
   17 WRITE(3,27)
   27 FORMAT('0  REAL JT    IMAG JT    REAL JP    IMAG JP    MAG  JT
     1MAG  JP')
      DO 10 J=1,NT
      P1=CABS(CUR(J))
      C2=CUR(J+NT)
      P2=CABS(C2)
      WRITE(3,11) CUR(J),C2,P1,P2
   11 FORMAT(1X,6E11.4)
   10 CONTINUE
      WRITE(3,12)
   12 FORMAT('0   REAL Q      IMAG Q      MAG  Q')
      DO 13 J=1,NT
      P1=CABS(Q(J))
      WRITE(3,11) Q(J),P1
   13 CONTINUE
      STOP
      END
$DATA
 31 0.1256637E-01
$STOP
/*
//
```

PRINTED OUTPUT

```
 NT      XC
 31 0.1256637E-01

  REAL JT    IMAG JT    REAL JP    IMAG JP    MAG  JT    MAG  JP
-0.1500E+01 0.2932E-01 0.1500E+01-0.2932E-01 0.1500E+01 0.1500E+01
-0.1500E+01 0.2916E-01 0.1492E+01-0.2909E-01 0.1500E+01 0.1492E+01
-0.1500E+01 0.2868E-01 0.1467E+01-0.2842E-01 0.1500E+01 0.1467E+01
-0.1500E+01 0.2789E-01 0.1426E+01-0.2732E-01 0.1500E+01 0.1427E+01
```

```
-0.1500E+01 0.2679E-01 0.1370E+01-0.2586E-01 0.1500E+01 0.1370E+01
-0.1500E+01 0.2539E-01 0.1299E+01-0.2409E-01 0.1500E+01 0.1299E+01
-0.1500E+01 0.2372E-01 0.1213E+01-0.2209E-01 0.1500E+01 0.1214E+01
-0.1500E+01 0.2179E-01 0.1115E+01-0.1994E-01 0.1500E+01 0.1115E+01
-0.1500E+01 0.1962E-01 0.1004E+01-0.1776E-01 0.1500E+01 0.1004E+01
-0.1500E+01 0.1723E-01 0.8816E+00-0.1561E-01 0.1500E+01 0.8817E+00
-0.1500E+01 0.1466E-01 0.7499E+00-0.1361E-01 0.1500E+01 0.7501E+00
-0.1500E+01 0.1193E-01 0.6101E+00-0.1184E-01 0.1500E+01 0.6102E+00
-0.1500E+01 0.9060E-02 0.4635E+00-0.1038E-01 0.1500E+01 0.4636E+00
-0.1500E+01 0.6096E-02 0.3118E+00-0.9284E-02 0.1500E+01 0.3120E+00
-0.1500E+01 0.3064E-02 0.1568E+00-0.8608E-02 0.1500E+01 0.1570E+00
-0.1500E+01-0.9828E-06 0.4459E-06-0.8379E-02 0.1500E+01 0.8379E-02
-0.1500E+01-0.3066E-02-0.1568E+00-0.8608E-02 0.1500E+01 0.1570E+00
-0.1500E+01-0.6098E-02-0.3118E+00-0.9285E-02 0.1500E+01 0.3120E+00
-0.1500E+01-0.9062E-02-0.4635E+00-0.1038E-01 0.1500E+01 0.4636E+00
-0.1500E+01-0.1193E-01-0.6101E+00-0.1184E-01 0.1500E+01 0.6102E+00
-0.1500E+01-0.1466E-01-0.7499E+00-0.1362E-01 0.1500E+01 0.7501E+00
-0.1500E+01-0.1724E-01-0.8816E+00-0.1562E-01 0.1500E+01 0.8817E+00
-0.1500E+01-0.1962E-01-0.1004E+01-0.1776E-01 0.1500E+01 0.1004E+01
-0.1500E+01-0.2179E-01-0.1115E+01-0.1995E-01 0.1500E+01 0.1115E+01
-0.1500E+01-0.2372E-01-0.1213E+01-0.2209E-01 0.1500E+01 0.1214E+01
-0.1500E+01-0.2540E-01-0.1299E+01-0.2409E-01 0.1500E+01 0.1299E+01
-0.1500E+01-0.2679E-01-0.1370E+01-0.2586E-01 0.1500E+01 0.1370E+01
-0.1500E+01-0.2789E-01-0.1426E+01-0.2732E-01 0.1500E+01 0.1427E+01
-0.1500E+01-0.2868E-01-0.1467E+01-0.2842E-01 0.1500E+01 0.1467E+01
-0.1500E+01-0.2916E-01-0.1492E+01-0.2909E-01 0.1500E+01 0.1492E+01
-0.1500E+01-0.2932E-01-0.1500E+01-0.2932E-01 0.1500E+01 0.1500E+01
```

```
   REAL Q      IMAG Q      MAG  Q
 0.1899E-01-0.1988E-03 0.1899E-01
 0.3136E+00-0.3256E-02 0.3136E+00
 0.6238E+00-0.6390E-02 0.6238E+00
 0.9271E+00-0.9234E-02 0.9271E+00
 0.1220E+01-0.1167E-01 0.1220E+01
 0.1500E+01-0.1361E-01 0.1500E+01
 0.1763E+01-0.1494E-01 0.1764E+01
 0.2008E+01-0.1562E-01 0.2008E+01
 0.2230E+01-0.1562E-01 0.2230E+01
 0.2427E+01-0.1494E-01 0.2427E+01
 0.2598E+01-0.1361E-01 0.2598E+01
 0.2741E+01-0.1168E-01 0.2741E+01
 0.2853E+01-0.9237E-02 0.2853E+01
 0.2935E+01-0.6393E-02 0.2935E+01
 0.2984E+01-0.3270E-02 0.2984E+01
 0.3000E+01-0.3979E-05 0.3000E+01
 0.2984E+01 0.3262E-02 0.2984E+01
 0.2935E+01 0.6385E-02 0.2935E+01
 0.2853E+01 0.9229E-02 0.2853E+01
 0.2741E+01 0.1167E-01 0.2741E+01
 0.2598E+01 0.1360E-01 0.2598E+01
 0.2427E+01 0.1494E-01 0.2427E+01
 0.2230E+01 0.1562E-01 0.2230E+01
 0.2008E+01 0.1562E-01 0.2008E+01
 0.1763E+01 0.1494E-01 0.1764E+01
 0.1500E+01 0.1360E-01 0.1500E+01
 0.1220E+01 0.1167E-01 0.1220E+01
 0.9271E+00 0.9232E-02 0.9271E+00
 0.6238E+00 0.6388E-02 0.6238E+00
 0.3136E+00 0.3265E-02 0.3136E+00
 0.1036E-02 0.1085E-04 0.1036E-02
```

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